The Image

Given two vector spaces V and W and a linear transformation $L : V \to W$ we define a set:

$$\mathrm{Im}(L) = \{\vec{w} \in W \mid \exists \vec{v} \in V, L(\vec{v}) = \vec{w}\} = L(V)$$

which we call the **image** of L. (sometimes called the **range** of L as a function)

Theorem

As defined above, the set $\mathrm{Im}(L)$ is a subspace of W, in particular it is a vector space.

Proof Sketch: We check the three conditions.

1. We know that $L(\vec{0}) = \vec{0}$, and so $\vec{0} \in \mathrm{Im}(L)$.
2. Let $\vec{w}_1, \vec{w}_2 \in \mathrm{Im}(L)$, then by definition there exists $\vec{v}_1, \vec{v}_2 \in V$ such that
$$L(\vec{v}_1) = \vec{w}_1 \qquad \text{and} \qquad L(\vec{v}_2) = \vec{w}_2$$
we then have that $\vec{v}_3 = \vec{v}_1 + \vec{v}_2 \in U$ and so
$$L(\vec{v}_3) = L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2) = \vec{w}_1 + \vec{w}_2$$
which shows that $\vec{w}_1 + \vec{w}_2 \in \mathrm{Im}(L)$.
3. Let $\vec{w} \in \mathrm{Im}(L)$,then by definition there exists $\vec{v} \in V$ such that
$$L(\vec{v}) = \vec{w}$$
now let $a \in \mathbb{R}$ be arbitrary, then we have $a\vec{v} \in V$ and so
$$L(a\vec{v}) = aL(\vec{v}) = a\vec{w}$$
which shows $a\vec{w} \in \mathrm{Im}(L)$.

Example - Images for Matricies

Describe and find a basis for, the image of the linear transformation, L, associated to

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

Well, things in the image look like

$$L((x,y,z)) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} + y \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + z \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

as we run over $x, y, z \in \mathbb{R}$, but that also describes

$$\mathrm{Span}((1,3,1),(2,2,1),(3,1,1))$$

So what is a basis? We find it by finding a basis of the span.

Theorem

If A is any matrix, then $\mathrm{Im}(A)$, or equivalently $\mathrm{Im}(L)$ for L the linear transformation associated to A, is precisely the span of the **columns** of A

Proof Idea: If A has columns $\vec{a}_1, \ldots, \vec{a}_\ell$ then

$$\mathrm{Im}(A) = \{A\vec{x} \mid \vec{x} \in \mathbb{R}^\ell\} = \{x_1\vec{a}_1 + \cdots + x_\ell\vec{a}_\ell \mid (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell\}$$

But this is precisely the span of $\vec{a}_1, \ldots, \vec{a}_\ell$.

This is a special case of a more general result we will see shortly.

Generating Sets and The Image.

Let V and W be a pair of vector spaces, suppose

$$L : V \to W$$

is a linear transformation.

Theorem

If $\vec{g}_1, \ldots, \vec{g}_n$ is a generating set for V then

$$L(\vec{g}_1), \ldots, L(\vec{g}_n)$$

is a generating set for $\mathrm{Im}(L)$.

This is on the assignment A3Q1(a).

Recall in the previous example that $(1,0,0), (0,1,0), (0,0,1)$ are a generating set for $\mathbb{R}^3$, and the columns of A are

$$L((1,0,0)), \quad L((0,1,0)), \quad L((0,0,1))$$

which were a generating set for the image.

The Rank of a Linear Transformation

We define the **Rank** of a linear transformation L to be the dimension of $\mathrm{Im}(L)$.

Recall that a function $L : V \to W$ is **surjective** if and only if

$$\forall \vec{w} \in W, \exists \vec{v} \in V, L(\vec{v}) = \vec{w}$$

We can rewrite this as saying, $L : V \to W$ is **surjective** if and only if

$$\forall \vec{w} \in W, \vec{w} \in \mathrm{Im}(L)$$

which is the same as

$$\mathrm{Im}(L) = W$$

Theorem

A linear transformation $L : V \to W$ is surjective if and only if $\mathrm{Im}(L) = W$.
If V is finite dimensional, this is if and only if $\mathrm{rank}(L) = \dim(W)$.

Proof Idea: We have just explained the first claim above, for the second recall that for finite dimensional vector spaces, if one is a subspace of the other, they are equal if and only if they have the same dimension.

This gives us a simple numerical characterization of surjectivity.

Generating Sets and Surjectivity.

Theorem
If $\vec{g}_1, \ldots, \vec{g}_n$ a generating set for V then L is surjective if and only if $L(\vec{g}_1), \ldots, L(\vec{g}_n)$ is a generating set for W.

Proof: We know

$$\mathrm{Im}(L) = W \Leftrightarrow L \text{ is surjective}$$

but we also know from the above that

$$\mathrm{Im}(L) = \mathrm{Span}(L(\vec{g}_1), \ldots, L(\vec{g}_n))$$

so the above becomes

$$\mathrm{Span}(L(\vec{g}_1), \ldots, L(\vec{g}_n)) = W \Leftrightarrow L \text{ is surjective}$$

which is the result.

Compositions of Surjective Maps

The following is actually a result about functions:

Theorem

if $f : A \to B$ and $g : B \to C$ are surjective functions, then $g \circ f : A \to C$ is surjective.

Theorem

if $f : A \to B$ and $g : B \to C$ are functions, and if $g \circ f : A \to C$ is surjective then g is surjective.

These theorems immediately implies the same for linear transformations.

Natural Questions About Images

- Find a generating set for the image.
  When dealing with $\mathbb{R}^n$ this is just the columns of the matrix.
- Find a basis for the image.
  do what you always do to find a basis. Use generators from V to get generators for the image, then find a linearly independent subset.
- Is the vector $\vec{v}$ in the image?
  This leads directly to a system of equations, solve $A\vec{x} = \vec{v}$.
- Is the map surjective?
  check that the rank = dimension or check the definition.