The kernel of a linear transformation is a vector subspace.

Given two vector spaces V and W and a linear transformation $L : V \to W$ we define a set:

$$\mathrm{Ker}(L) = \{\vec{v} \in V \mid L(\vec{v}) = \vec{0}\} = L^{-1}(\{\vec{0}\})$$

which we call the **kernel** of L. (some people call this the **nullspace** of L).

Theorem
As defined above, the set $\mathrm{Ker}(L)$ is a subspace of V, in particular it is a vector space.
Proof Sketch We check the three conditions

1. Because we know $L(\vec{0}) = \vec{0}$ we know $\vec{0} \in \mathrm{Ker}(L)$.
2. Let $\vec{v}_1, \vec{v}_2 \in \mathrm{Ker}(L)$ then we know

$$L(\vec{v}_1 + \vec{v}_2) = L(\vec{v}_1) + L(\vec{v}_2) = \vec{0} + \vec{0} = \vec{0}$$

   and so $\vec{v}_1 + \vec{v}_2 \in \mathrm{Ker}(L)$.
3. Let $\vec{v} \in \mathrm{Ker}(L)$ and $a \in \mathbb{R}$ then

$$L(a\vec{v}) = aL(\vec{v}) = a\vec{0} = \vec{0}$$

   and so $a\vec{v} \in \mathrm{Ker}(L)$.

Example - Kernels Matricies

Describe and find a basis for the kernel, of the linear transformation, L, associated to

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

The kernel is precisely the set of vectors (x, y, z) such that $L((x, y, z)) = (0, 0, 0)$, so

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

but this is precisely the solutions to the system of equations given by A!

So we find a basis by solving the system!

Theorem

If A is any matrix, then $\mathrm{Ker}(A)$, or equivalently $\mathrm{Ker}(L)$, where L is the associated linear transformation, is precisely the solutions $\vec{x}$ to the system

$$A\vec{x} = \vec{0}$$

This is immediate from the definition given our understanding of how to associate a system of equations to

$$M\vec{x} = \vec{0}.$$

The Kernel and Injectivity

Recall that a function $L : V \to W$ is injective if

$$\forall \vec{v}_1, \vec{v}_2 \in V, ((L(\vec{v}_1) = L(\vec{v}_2)) \Rightarrow (\vec{v}_1 = \vec{v}_2))$$

Theorem

A linear transformation $L : V \to W$ is injective if and only if $\mathrm{Ker}(L) = \{\vec{0}\}$.

Proof:

$\Rightarrow$-direction We assume that L is injective.

Let $\vec{v} \in \mathrm{Ker}(L)$, then

$$L(\vec{v}) = \vec{0} = L(\vec{0})$$

since L is injective this implies $\vec{v} = \vec{0}$, from which we conclude $\mathrm{Ker}(L) = \{\vec{0}\}$.

$\Leftarrow$-direction We assume $\mathrm{Ker}(L) = \{\vec{0}\}$.

Let $\vec{v}_1, \vec{v}_2 \in V$ be arbitrary and assume $L(\vec{v}_1) = L(\vec{v}_2)$.

We then have that

$$L(\vec{v}_1) - L(\vec{v}_2) = \vec{0} \qquad \Rightarrow \qquad L(\vec{v}_1 - \vec{v}_2) = \vec{0}$$

and so $\vec{v}_1 - \vec{v}_2 \in \mathrm{Ker}(L) = \{\vec{0}\}$ and so

$$\vec{v}_1 - \vec{v}_2 = \vec{0} \qquad \Rightarrow \qquad \vec{v}_1 = \vec{v}_2$$

which proves L is injective.

The Nullity of a Linear Transformation

We define the **Nullity** of a linear transformation L to be the dimension of $\mathrm{Ker}(L)$.

Theorem

A linear transformation $L : V \to W$ is injective if and only if $\mathrm{Ker}(L) = \{\vec{0}\}$.
In particular this is if and only if $\mathrm{null}(L) = 0$.

Proof:

We just proved the first claim, the second is an immediate consequence because we know

$$\{\vec{0}\} \subset \mathrm{Ker}(L)$$

and so they are equal only if they have the same dimension, that is 0.

Linear Independence and Injectivity

$L : V \to W$ is a linear tranformation, $\vec{s}_1, \dots, \vec{s}_n$ is a collection of vectors in V and hence $L(\vec{s}_1), \dots, L(\vec{s}_n)$ is a collection of vectors in W.

Theorem

If $\vec{s}_1, \dots, \vec{s}_n$ are a basis for V, and L is not injective, then $L(\vec{s}_1), \dots, L(\vec{s}_n)$ are linearly dependent.

Proof Sketch As L is not injective there is $\vec{0} \neq \vec{v} \in \mathrm{Ker}(L)$.
Because $\vec{s}_1, \dots, \vec{s}_n$ are a basis for V we can find $a_1, \dots, a_n \in \mathbb{R}$ so that

$$\vec{v} = a_1\vec{v}_1 + \cdots + a_n\vec{v}_n$$

and as $\vec{v} \neq \vec{0}$, at least one of the $a_i \neq 0$. But then we have

$$\vec{0} = L(\vec{v}) = L(a_1\vec{v}_1 + \cdots + a_n\vec{v}_n) = a_1L(\vec{v}_1) + \cdots + a_nL(\vec{v}_n)$$

which gives a non-trivial relation on $L(\vec{s}_1), \dots, L(\vec{s}_n)$ and so they are linearly dependent.

Theorem

If $L(\vec{s}_1), \dots, L(\vec{s}_n)$ are linearly independent then $\vec{s}_1, \dots, \vec{s}_n$ are linearly independent.

On the assignment A3Q1(b)

Theorem

If $\vec{s}_1, \dots, \vec{s}_n$ are linearly independent, and L is injective, then $L(\vec{s}_1), \dots, L(\vec{s}_n)$ are linearly independent.

Direct Consequence of A3Q1(c), using that L is injective implies $\mathrm{Ker}(L) = \{\vec{0}\}$

Compositions of Injective Maps

The following is actually a result about functions:

Theorem

if $f : A \to B$ and $g : B \to C$ are injective functions, then $g \circ f : A \to C$ is injective.

Theorem

if $f : A \to B$ and $g : B \to C$ are functions, and if $g \circ f : A \to C$ is injective then f is injective.

These theorems immediately implies the same for linear transformations.

Natural Questions About Images

- Find a basis for the kernel.
  For $\mathbb{R}^n$ this is just solving the system for the associated matrix.
- Find the dimension of the kernel.
  Typically find a basis.
- Is the vector $\vec{v}$ in the kernel?
  Check the definition.
- Is the map injective?
  Check if the kernel is $\{\vec{0}\}$.