Which Linear Transformations are Invertible

We have mentioned taking inverses of linear transformations. But when can we do this?

Theorem

A linear transformation is invertible if and only if it is injective and surjective.

This is a theorem about functions.

Theorem

A linear transformation $L : U \to V$ is invertible if and only if $\ker(L) = \{\vec{0}\}$ and $\mathrm{Im}(L) = V$.

This follows from our characterizations of injective and surjective.

Theorem

A linear transformation $L : U \to V$ is invertible if and only if whenever $\vec{e}_1, \dots, \vec{e}_n$ is a basis for U the collection $L(\vec{e}_1), \dots, L(\vec{e}_n)$ is a basis for V.

Proof: $\Rightarrow$-direction. We assume L is bijective.

Then L is injective, so $\mathrm{Ker}(L) = \{\vec{0}\}$ and so

$$\mathrm{Ker}(L) \cap \mathrm{Span}(\vec{e}_1, \dots, \vec{e}_n) = \{\vec{0}\}$$

so by the assignment, $L(\vec{e}_1), \dots, L(\vec{e}_n)$ are linearly independent.

Because L is surjective we know $\mathrm{Im}(L) = V$, and as $\vec{e}_1, \dots, \vec{e}_n$ are a basis for U they are a generating set, and so so by the assignment, $L(\vec{e}_1), \dots, L(\vec{e}_n)$ are a generating set for $\mathrm{Im}(L) = V$.

We conclude $L(\vec{e}_1), \dots, L(\vec{e}_n)$ are a basis.

$\Leftarrow$-direction this is on the assignment

Finite Dimensional Case

Theorem (Rank-Nullity Theorem)
Suppose $L : U \to V$ is a linear transformation between finite dimensional vector spaces then $\mathrm{null}(L) + \mathrm{rank}(L) = \dim(U)$.
We will eventually give two (different) proofs of this.

Theorem
Suppose U and V are finite dimensional vector spaces a linear transformation $L : U \to V$ is invertible if and only if $\mathrm{rank}(L) = \dim(V)$ and $\mathrm{null}(L) = 0$.
Proof Idea This is just checking surjectivity and injectivity by looking at the dimensions of the image and kernel.

Theorem
Suppose U and V are finite dimensional vector spaces a linear transformation $L : U \to V$ is invertible if and only if $\dim(U) = \dim(V)$ and $\mathrm{rank}(L) = \dim(V)$.

Theorem
Suppose U and V are finite dimensional vector spaces a linear transformation $L : U \to V$ is invertible if and only if $\dim(U) = \dim(V)$ and $\mathrm{null}(L) = 0$.

Proof Idea These last two results just playing games with the equalities in the above theorems.

Finite Dimensional Case - Matrix

Recall: The following result just says that we can check invertibility by looking at the matrix.

Theorem

Suppose U and V are finite dimensional vector spaces a linear transformation $L : U \to V$ is invertible if and only if either equivalently

- For some choice of basis for U and V the matrix associated to L is invertible.
- For any choice of basis for U and V the matrix associated to L is invertible.

Proof

$\Rightarrow$-direction assuming L invertible let M be its inverse, then we have the formulas

$$L \circ M = \mathrm{Id}_V \qquad \text{and} \qquad M \circ L = \mathrm{Id}_U$$

thus for any choice of basis, if A is the matrix for L and B is the matrix for M we know that

$$AB = \mathrm{Id} \qquad \text{and} \qquad BA = \mathrm{Id}$$

because the matrix for Id_V and Id_U are always the identity matrix.
This proves the matricies are always invertible.
$\Leftarrow$-direction Fix any basis in which the matrix A, associated to L is invertible.
In the same basis, let M be the matrix associated to A^{-1}.
Then $L \circ M$ and $M \circ L$ are respectively the transformations associated to

$$AA^{-1} = \mathrm{Id} \qquad \text{and} \qquad A^{-1}A = \mathrm{Id}$$

Invertibility of a Matrix

Theorem

A (sqaure) matrix A is invertible if and only if the determinant is non-zero.

There are lots of different ways to prove this, depending on what you know about determinants.

For some other approaches see the notes on the determinant on Moodle or check in your textbook.

If the determinant is non-zero then we can check directly that

$$\left(\frac{1}{\det(A)}\mathrm{Adj}(A)\right)A = \mathrm{Id} = A\left(\frac{1}{\det(A)}\mathrm{Adj}(A)\right)$$

by using the definition of $\mathrm{Adj}(A)$ (if you forget what this is ask me about it later, we will never use it for anything else) and properties of the determinant.

Conversely if A has an inverse then by multiplicativity of the determinant

$$\det(A)\det(A^{-1}) = \det(AA^{-1}) = \det(\mathrm{Id}) = 1$$

and so if there is an inverse, the determinant can't be zero.

Invertibility of a Matrix - Other Characterizations

Theorem

Suppose A is an n by n (so square) matrix then the following are equivalent:

1. A is invertible.
2. $\det(A)$ is non-zero. See previous slide
3. A^t is invertible. on assignment 1
4. The reduced row echelon form of A is the identity matrix. (algorithm to find inverse)
5. A has rank n, rank is number of lead 1s in RREF
6. the columns of A span $\mathbb{R}^n$, rank is dim of span of columns
7. $A\vec{x} = \vec{b}$ always has a solution, definition of columns spanning
8. the columns of A are a basis for $\mathbb{R}^n$, generating set of size n must be LI
9. the columns of A are linearly independent. basis is LI
10. when $A\vec{x} = \vec{b}$ has a solution it is unique, LI implies unique representations
11. The kernel of A is $\{\vec{0}\}$ What you check when you check LI
12. A has nullity 0, Definition of nullity
13. the rows of A span $\mathbb{R}^n$, Apply above to A^t
14. the rows of A are a basis for $\mathbb{R}^n$, Apply above to A^t
15. the rows of A are linearly independent. Apply above to A^t

Isomorphisms

Recall

We call an invertible linear transformation between vector spaces U and V an **isomorphism**.

We say that vector spaces are U and V are **isomorphic** if there exists an isomorphism between them, so if there exists a bijective $L : U \to V$.

Theorem

Vector spaces U and V are isomorphic if and only if $\dim(U) = \dim(V)$.

Proof:

$\Rightarrow$-direction recall that if L is bijective, and B is a basis for U, then $L(B)$ is a basis for V, hence both have a basis of the same size.

$\Leftarrow$-direction

If $\{\vec{e}_i\}$ is a basis for U and $\{\vec{f}_i\}$ is a basis for V, and both have the same size then we can define maps

$$L : U \to V \qquad \text{and} \qquad M : V \to U$$

by

$$L(\vec{e}_i) = \vec{f}_i \qquad \text{and} \qquad M(\vec{f}_i) = \vec{e}_i$$

notice why we need the bases to have the same size

It is clear that these maps are inverses, thus give the desired isomorphism.

Natural Questions About Isomorphisms and Inverses

- Given some description of a linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$, is it an isomorphism? does it have an inverse? and if yes, what is a description for the inverse?
  There are a lot of conditions you could check, and it is not always obvious which one is easiest. To find the inverse you pretty much always use gaussian elimination.
- Given some description of a linear transformation $L : V \to W$, is it an isomorphism? does it have an inverse? and if yes, what is a description for the inverse?
  There are a lot of conditions you could check, and it is not always obvious which one is easiest.