Matrices Built Around Invariant Subspaces

We say a subspace V_1 of V is **L-invariant**, with respect to a linear transformation $L : V \to V$ if $L(V_1) \subset V_1$.

Suppose $L : V \to V$ is a linear transformation of a finite dimensional vector space and

$$V = V_1 \oplus V_2$$

is a decomposition with V_1 being L-invariant (V_2 need not be).
Pick basis $\vec{e}_1, \ldots, \vec{e}_\ell$ for V_1 and $\vec{f}_1, \ldots, \vec{f}_m$ for V_2.
Recall we previously defined the induced maps:

$$L_{11} : V_1 \to V_1,\ L_{21} : V_1 \to V_2,\ L_{12} : V_2 \to V_1,\ L_{22} : V_2 \to V_2$$

and we denote by $A_{11}, A_{12}, A_{21}, A_{22}$ the respective matricies in this context.

Proposition

If V_1 is L-invariant then $L_{21} = 0$ is the zero map and hence $A_{21} = 0$ and the matrix for L with respect to this basis is:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

This is immediate from how we construct the matrix for a block decomposition. We leave the details as an exercise.

Characteristic Polynomials

Lemma
Suppose $L : V \to V$ is a linear transformation of a finite dimensional vector space and

$$V = V_1 \oplus V_2$$

is a decomposition with V_1 being L-invariant (V_2 need not be).
Denote by $L_i : V_i \to V_i$ the map induced by L. We have that:

$$\mathrm{char}_L(x) = \prod_i \mathrm{char}_{L_i}(x)$$

This follows directly from the fact that

$$\mathrm{Det}\left(\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\right) = \mathrm{Det}(A)\mathrm{Det}(C)$$

which is a bonus question on the assignment.
warning the proof of the statement about determinants is a bit annoying notationally

Minimal Polynomials

Lemma

Suppose $L : V \to V$ is a linear transformation of a finite dimensional vector space and

$$V = V_1 \oplus V_2$$

is a decomposition with V_1 being L-invariant (V_2 need not be).
Denote by $L_i : V_i \to V_i$ the map induced by L. We have that:

$$\mathrm{LCM}(\min_{L_i}(x)) | \min_L(x)$$

Proof Notice that for any polynomial P we have

$$P\left(\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}\right) = \begin{pmatrix} P(A_{11}) & ?? \\ 0 & P(A_{22}) \end{pmatrix}$$

It follows that because $\min_{\mathrm{A}}(A) = 0$ then $\min_{\mathrm{A}}(A_{11}) = 0$ and $\min_{\mathrm{A}}(A_{22}) = 0$.
Next recall that for any polynomial P, and matrix B that $P(B) = 0$ implies $\min_B(x)|P(x)$, so in particular by the above we have $\min_{A_{11}}(x)|\min_{\mathrm{A}}(x)$ and $\min_{A_{22}}(x)|\min_{\mathrm{A}}(x)$.
From the definition of LCM because $\min_{A_{11}}(x)|\min_{\mathrm{A}}(x)$ and $\min_{A_{22}}(x)|\min_{\mathrm{A}}(x)$ we know $\mathrm{LCM}(\min_{L_1}(x), \min_{L_2}(x))|\min_L(x)$.
We note that we only get divisibility on the minimal polynomial because we do not know if the top right corner is 0!

Example

If $L : V \to V$ is a linear transformation and $\vec{0} \neq \vec{v} \in V$ is a vector such that

$$L(\vec{v}) = \lambda \vec{v}$$

Then the subspace $\mathrm{Span}(\vec{v})$ is L-invariant, and hence with $V_1 = \mathrm{Span}(\vec{v})$ and V_2 any complementary subspace we can find some basis for which the matrix is of the form:

$$\begin{pmatrix} \lambda & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

Theorem (we won't use this for anything)
For every linear transformation $L : V \to V$ there exists a basis in which the matrix is upper triangular.
Proof Idea The idea is by induction using the idea above.
Assume any $n-1$ by $n-1$ matrix can be put into upper triangular form.
Consider A an n by n matrix.
Picking any root λ of the characteristic polynomial of A we can find an eigenvector and do the above.
The inductive hypothesis now says we can pick a basis for V_2 such that A_{22} is upper triangular.
This gives the result.

Matrices Built Around Invariant Direct Sum Decompositions

Suppose $V = V_1 \oplus V_2$, we say this is an L-**invariant decomposition**, with respect to a linear transformation $L : V \to V$, if both of V_1 and V_2 are L-invariant.

Pick basis $\vec{e}_1, \dots, \vec{e}_\ell$ for V_1 and $\vec{f}_1, \dots, \vec{f}_m$ for V_2.
As before we have

$$L_{11} : V_1 \to V_1,\ L_{21} : V_1 \to V_2,\ L_{12} : V_2 \to V_1,\ L_{22} : V_2 \to V_2$$

and we denote by $A_{11}, A_{12}, A_{21}, A_{22}$ the respective matricies in this context.
Theorem
If $V = V_1 \oplus V_2$ is an L-invariant decomposition then both L_{21} and L_{12} are the zero maps and hence $A_{21} = 0$ and $A_{12} = 0$ and the matrix for L with respect to this basis is:

$$A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$$

Having both spaces be invariant is exactly the condition that guarentees the upper right and lower left blocks will be 0.

Characteristic and Minimal Polynomials

Theorem

Suppose $L : V \to V$ is a linear transformation of a finite dimensional vector space and

$$V = V_1 \oplus \cdots \oplus V_r$$

is a decomposition into L invariant subspaces.
Denote by $L_i : V_i \to V_i$ the map induced by L. We have that:

$$\mathrm{char}_L(x) = \prod_i \mathrm{char}_{L_i}(x) \qquad \mathrm{min}_L(x) = \mathrm{LCM}(\mathrm{min}_{L_i}(x))$$

Proof: The claim for $\mathrm{char}_L(x)$ is immediate from the invariant subspace case.
We know that there are polynomials $P_1(x)$ and $P_2(x)$ so that

$$Q(x) := \mathrm{LCM}(\mathrm{min}_{L_1}(x), \mathrm{min}_{L_2}(x)) = \mathrm{min}_{L_1}(x)P_1(x) = \mathrm{min}_{L_2}(x)P_2(x)$$

Thus $Q(A_{11}) = \mathrm{min}_{L_1}(A_{11})P(A_{11}) = 0P_1(A_{11}) = 0$ and similarly $Q(A_{22}) = 0$.
From this we conclude that

$$Q(A) = 0$$

and thus that

$$\mathrm{min}_L(x) \Big| Q(x) = \mathrm{LCM}(\mathrm{min}_{L_1}(x), \mathrm{min}_{L_2}(x)).$$

Combined with the fact that we already know $\mathrm{LCM}(\mathrm{min}_{L_1}(x), \mathrm{min}_{L_2}(x)) \Big| \mathrm{min}_L(x)$ we get the equality.

Recursive Strategy For Finding Good Basis

- Find an invariant direct sum decompositon $V = V_1 \oplus V_2$.
- Rewrite the matrix in that basis:

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

- Repeat the process seperately for A and B (study V_1 and V_2 on their own). Noting that any change of basis involving only V_1 only effects A and not B and vice versa.
- Once we can no longer cut things into smaller pieces, find nice descriptions for each piece seperately.

This idea is ultimately why we can eventually get matricies into the form:

$$A = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{pmatrix}$$

At which point we only need to worry about making the A_i look pretty.

Some Theorems About Invariant Subspaces

Lemma

If U_1 and U_2 are L-invariant subspaces, then $U_1 \cap U_2$ is L-invariant.

Proof:

Note To show a subset U is L-invariant we need to show $L(U) \subset U$, which means we must check

$$\forall \vec{v} \in L(U), \vec{v} \in U$$

but $\vec{v} \in L(U)$ means $\vec{v} = L(\vec{u})$ for $\vec{u} \in U$, so we check

$$\forall \vec{u} \in U, L(\vec{u}) \in U$$

Let $\vec{u} \in U_1 \cap U_2$ be arbitrary, we must prove that $L(\vec{u}) \in U$.
Since $\vec{u} \in U_1 \cap U_2$ we know $\vec{u} \in U_1$, and since U_1 is L-invariant this means $L(\vec{u}) \in U_1$.
Since $\vec{u} \in U_1 \cap U_2$ we know $\vec{u} \in U_2$, and since U_2 is L-invariant this means $L(\vec{u}) \in U_2$.
Since $L(\vec{u}) \in U_1$ and $L(\vec{u}) \in U_2$ we know $L(\vec{u}) \in U_1 \cap U_2$, which gives the result.

Lemma

The subspaces $\mathrm{Ker}(L)$ and $\mathrm{Im}(L)$ are always L-invariant.

This is a direct consequence of an assignment question

Some **NOT** Theorems about Invariant Subspaces

Many things we wish were true about invariant subspaces sadly are not, here are some examples:

- Not every invariant subspace is part of an invariant direct sum decomposition. You are asked for an example of this on the assignment.
- If $U \subset V$ is L-invariant, and $W \subset U$, the subspace W doesn't need to be L-invariant.

Natural Questions About Invariant Subspaces

- Is this subspace invariant?
  Check the definition, perhaps just by chiecking on a basis
- Is there an invariant complementary subspace?
  This question is hard in general, we won't talk about it.
- Is this direct sum decomposition invariant?
  Check by evaluating on a basis
- Find a non-trivial invariant subspace.
  We will explain a few ways to do this.
- Find a non-trivial invariant direct sum decomposition.
  We will explain basically one way to do this.