Reminders

April 2, 2020 10:41 AM

Today: eigenvalues, eigenvectors + Review

Vectors

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Add: $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -1 \end{bmatrix}$

Scale: $5\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ -5 \\ 15 \end{bmatrix}$

$\|c\vec{x}\| = |c|\|\vec{x}\|$

length: $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}$

$\|\vec{x}\| = \sqrt{1^2 + 2^2 + 4^2}$

$= \sqrt{21}$

unit: $\|\vec{u}\| = 1$

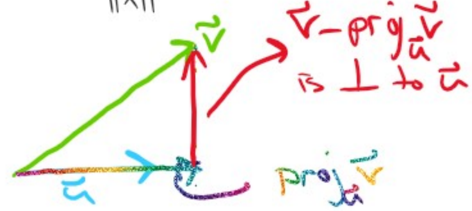
$\vec{u} = \frac{1}{\|\vec{x}\|}\vec{x}$

Dot: $\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = 1(3) + 3(1) + (-2)(4)$

$= 3 + 3 - 8 = -2$

$\vec{u} \cdot \vec{v} = \|\vec{u}\|\|\vec{v}\| \cos\theta$

$\text{proj}_{\vec{u}} \vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right)\vec{u}$



Given $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 1 & -1 & 4 \end{bmatrix}$ $\quad |A| = 2\begin{vmatrix} 0 & 2 \\ -1 & 4 \end{vmatrix} - 1\begin{vmatrix} -1 & 2 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 0 \\ 1 & -1 \end{vmatrix} = 2(2) - 1(-6) + 3(1)$

$= 13$

Given $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ $\quad \vec{w} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$ $\quad \vec{v} \times \vec{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 1 \\ 2 & 2 & 3 \end{vmatrix} = \hat{i}\begin{vmatrix} -2 & 1 \\ 2 & 3 \end{vmatrix} - \hat{j}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + \hat{k}\begin{vmatrix} 1 & -2 \\ 2 & 2 \end{vmatrix}$

$= \hat{i}(-8) - \hat{j}(1) + \hat{k}(6)$

$= \begin{bmatrix} -8 \\ -1 \\ 6 \end{bmatrix}$

Example 1

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$$A\vec{x} = \lambda\vec{x} \Rightarrow \boxed{(A-\lambda I)\vec{x} = \vec{0}}$$

homogeneous system

- want $\vec{x} \neq \vec{0}$.

$\to$ need $\det(A-\lambda I) = 0$.

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{bmatrix}$$

$$C_A(x) = \det(xI - A)$$

(you can use $\det(A - xI)$)

$$= \begin{vmatrix} x-2 & 0 & -2 \\ 0 & x-2 & 1 \\ -1 & 1 & x-4 \end{vmatrix} = (x-2)\begin{vmatrix} x-2 & 1 \\ 1 & x-4 \end{vmatrix} + 0 - 2\begin{vmatrix} 0 & x-2 \\ -1 & 1 \end{vmatrix}$$

$$= (x-2)\left((x-2)(x-4) - 1\right) - 2\left(0 + (x-2)\right)$$

$$= (x-2)[x^2 - 6x + \underbrace{8 - 1 - 2}_{5}]$$

$$= (x-2)(x-1)(x-5)$$

$C_A(\lambda) = 0$ for $\lambda = 1, 2, 5$.

$\lambda = 1$

$$A - \lambda I = A - I = \begin{bmatrix} 2-1 & 0 & 2 \\ 0 & 2-1 & -1 \\ 1 & -1 & 4-1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}$$

$(A - I)\vec{x} = \vec{0}$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 3 & 0 \end{array}\right] \xrightarrow{R_3 - R_1 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 + R_2 \to R_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 + 2x_3 = 0 \Rightarrow x_1 = -2x_3$

$x_2 - x_3 = 0$ $\quad x_2 = x_3$

x_3 is free.

$$\vec{x} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3\boxed{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$$

$$\boxed{\vec{x} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$$

$$\|\vec{x}\| = \sqrt{(-2)^2 + 1^2 + 1^2} = \sqrt{6}$$

$\lambda = 2$

$$A - 2I = \begin{bmatrix} 2-2 & 0 & 2 \\ 0 & 2-2 & -1 \\ 1 & -1 & 4-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$\boxed{\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

$$\xrightarrow{RREF} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$y_1 - y_2 = 0$

$y_3 = 0$

$$\vec{y} = \begin{bmatrix} y_2 \\ y_2 \\ 0 \end{bmatrix} = y_2\boxed{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} y_1 - y_2 &= 0 \\ y_3 &= 0 \\ y_2 &\text{ is free} \end{aligned} \qquad \vec{y} = \begin{bmatrix} y_2 \\ y_2 \\ 0 \end{bmatrix} = y_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$\lambda = 5$

$$A - 5I = \begin{bmatrix} 2-5 & 0 & 2 \\ 0 & 2-5 & -1 \\ 1 & -1 & 4-5 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 2 \\ 0 & -3 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -3 & -1 \\ -3 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 + 3R_1 \to R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \end{bmatrix}$$

$$\xrightarrow[\left(-\frac{1}{3}R_2 \to R_2\right)]{R_3 - R_2 \to R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2 \to R_1} \begin{bmatrix} 1 & 0 & -2/3 \\ 0 & 1 & 1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \qquad (A - 5I)\vec{z} = \vec{0} \implies \begin{aligned} z_1 - \tfrac{2}{3} z_3 &= 0 \\ z_2 + \tfrac{1}{3} z_3 &= 0 \end{aligned}$$

$$\begin{aligned} z_1 &= \tfrac{2}{3} z_3 \\ z_2 &= -\tfrac{1}{3} z_3 \\ z_3 &\text{ is free} \end{aligned} \qquad \text{choose } z_3 = 3 \implies \vec{z} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Note: Let $P = [\vec{x} \mid \vec{y} \mid \vec{z}] = \begin{bmatrix} -2 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix}$

$$AP = [A\vec{x} \mid A\vec{y} \mid A\vec{z}]$$
$$= [\vec{x} \mid 2\vec{y} \mid 5\vec{z}]$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 1 & 1 & -1 \\ 1 & 0 & 3 \end{bmatrix} = \begin{array}{ccc} \begin{bmatrix} -2 & 2 & 10 \\ 1 & 2 & -5 \\ 1 & 0 & 15 \end{bmatrix} \\ \vec{x} \quad 2\vec{y} \quad 5\vec{z} \end{array}$$

Exercise: 1. Find P^{-1}.

2. Confirm

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Example 3

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$$A = \begin{bmatrix} 2 & 0 & 0 \\ -2 & -2 & 2 \\ -5 & -10 & 7 \end{bmatrix}$$

$$C_A(x) = \begin{vmatrix} x-2 & 0 & 0 \\ 2 & x+2 & -2 \\ 5 & 10 & x-7 \end{vmatrix}$$

$$= (x-2)\begin{vmatrix} x+2 & -2 \\ 10 & x-7 \end{vmatrix}$$

$$= (x-2)(x^2-5x-14+20)$$

$$= (x-2)(x^2-5x+6)$$

$$= (x-2)(x-2)(x-3)$$

Eigenvalues:

$\lambda=2$ (multiplicity 2)

$\lambda=3$

$\boxed{\lambda=2}$

$$A-2I = \begin{bmatrix} 2-2 & 0 & 0 \\ -2 & -2-2 & 2 \\ -5 & -10 & 7-2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -2 & -4 & 2 \\ -5 & -10 & 5 \end{bmatrix}$$

$$\xrightarrow{RREF} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \text{ and } (A-2I)\vec{X}=\vec{0}$$

$$x_1+2x_2-x_3=0$$

$$\therefore \vec{X} = \begin{bmatrix} -2x_2+x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\boxed{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}} + x_3\boxed{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}$$

2 eigenvectors with $\lambda=2$: $\vec{x}_1$ $\vec{x}_2$

$\boxed{\lambda=3}$

$$A-3I = \begin{bmatrix} 2-3 & 0 & 0 \\ -2 & -2-3 & 2 \\ -5 & -10 & 7-3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ -2 & -5 & 2 \\ -5 & -10 & 4 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & 2 \\ 0 & -10 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2/5 \\ 0 & 0 & 0 \end{bmatrix}$$

$\vec{Y} = \begin{bmatrix} y_1 \\ \end{bmatrix}$ ... $(A-3I)\vec{Y}=\vec{0}$ $\Rightarrow y_1=0$

$$\begin{bmatrix} 0 & -10 & 4 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ and } (A - 3I)\vec{y} = \vec{0} \implies \begin{aligned} y_1 &= 0 \\ y_2 &= \tfrac{2}{5} y_3 \end{aligned}$$

– take $y_3 = 5$: $\vec{y} = \begin{bmatrix} 0 \\ 2 \\ 5 \end{bmatrix}$

Diagonalize: $P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$

Check: $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Example 4

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$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Note that A is symmetric

$A^T = A$

$A\vec{x} = \lambda\vec{x}$, $\vec{x} \neq \vec{0}$

$$C_A(x) = |xI - A|$$

$$= \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 1 \\ -1 & 1 & x-2 \end{vmatrix} = (x-1)\begin{vmatrix} x-1 & 1 \\ 1 & x-2 \end{vmatrix} - 1\begin{vmatrix} 0 & x-1 \\ -1 & 1 \end{vmatrix}$$

$$= (x-1)\left((x-1)(x-2)-1\right) - 1(x-1)$$

$$= (x-1)\left(x^2 - 3x + 2 - 1 - 1\right)$$

$$= (x-1)(x^2 - 3x) = x(x-1)(x-3)$$

$C_A(0) = |-A| = 0$

so $|A| = 0$

$\lambda = 0$, $\lambda = 1$, $\lambda = 3$.

$\lambda = 0$ $A - 0I = A$ $\Rightarrow$ solve $A\vec{x} = \vec{0}$ (A is not invertible)

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{R_3 - R_1 \to R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 + R_2 \to R_3} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$\therefore A\vec{x} = \vec{0}$ and $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow$ $x_1 = -x_3$, $x_2 = x_3$, x_3 is free

$x_3 = 1$ gives $\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = 1$ $A - 1I = \begin{bmatrix} 1-1 & 0 & 1 \\ 0 & 1-1 & -1 \\ 1 & -1 & 2-1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

$(A - I)\vec{y} = \vec{0}$, $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \Rightarrow$ $y_1 = y_2$, $y_3 = 0$

$y_2 = 1$ gives $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = 3$ $A - 3I = \begin{bmatrix} 1-3 & 0 & 1 \\ 0 & 1-3 & -1 \\ 1 & -1 & 2-3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & -2 & -1 \\ 1 & -1 & -1 \end{bmatrix}$

$$\xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & -1 \\ -2 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 + 2R_1 \to R_3} \begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow{R_3 - R_2 \to R_3}$$

$$\rightarrow \begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & -2 & -1 \\ 0 & -2 & -1 \end{bmatrix} \longrightarrow$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_2 \to R_2} \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 + R_2 \to R_1} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$(A - 3I)\vec{z} = \vec{0}$, $\vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$ $z_1 = \frac{1}{2} z_3$, $z_2 = -\frac{1}{2} z_3$, z_3 is free. Let $z_3 = 2$: $\vec{z} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Recap: $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ Eigenvalues: $\lambda = 0$ $\lambda = 1$ $\lambda = 3$

Eigenvectors: $\vec{x} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\vec{z} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Note: $\vec{x} \cdot \vec{y} = -1 + 1 + 0 = 0$
$\vec{x} \cdot \vec{z} = -1 - 1 + 2 = 0$
$\vec{y} \cdot \vec{z} = 1 - 1 + 0 = 0$

$\{\vec{x}, \vec{y}, \vec{z}\}$ is orthogonal!

Eg: $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \underset{4/3}{\cancel{c_1}} \vec{x} + \underset{3/2}{\cancel{c_2}} \vec{y} + \underset{5/6}{\cancel{c_3}} \vec{z}$ ~~$P = \begin{bmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix}$~~

$\vec{x} \cdot \vec{v} = \vec{x} \cdot (c_1 \vec{x} + c_2 \vec{y} + c_3 \vec{z}) = c_1(\vec{x} \cdot \vec{x}) + c_2(\vec{x} \cdot \vec{y}) + c_3(\vec{x} \cdot \vec{z})$

$4 = c_1(3) + 0 + 0 \Rightarrow c_1 = \frac{4}{3}$.

$c_2 = \dfrac{\vec{y} \cdot \vec{v}}{\vec{y} \cdot \vec{y}} = \dfrac{3}{2}$ $c_3 = \dfrac{\vec{z} \cdot \vec{v}}{\vec{z} \cdot \vec{z}} = \dfrac{5}{6}$

Normalize! $\vec{a} = \dfrac{1}{\|\vec{x}\|} \vec{x} = \dfrac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $\vec{b} = \dfrac{1}{\|\vec{y}\|} \vec{y} = \dfrac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\vec{c} = \dfrac{1}{\|\vec{z}\|} \vec{z} = \dfrac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$$P = [\vec{a} \ \vec{b} \ \vec{c}] = \begin{bmatrix} -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}$$

Check: $P^T = P^{-1}$

Check: 1. $P^T = P^{-1}$

2. $P^T A P = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}}_{D}$

$$A^7 = (PDP^T)^7 = PD^7P^T$$

$$D^7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^7 \end{bmatrix}$$

Example 5

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$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Notice:

$A^T = A$

$$C_A(x) = \begin{vmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{vmatrix}$$

$$= x\begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & x \end{vmatrix} - 1\begin{vmatrix} -1 & x \\ -1 & -1 \end{vmatrix}$$

$$= x(x^2-1) + (-x-1) - (1+x)$$

$$= x(x-1)(x+1) - 2(x+1)$$

$$= (x+1)(x(x-1)-2)$$

$$= (x+1)(x^2-x-2)$$

$$= (x+1)^2(x-2)$$

Eigenvalues:

$\lambda = -1$ (mult. 2)

$\lambda = 2$

$\lambda = 2$

$$A - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1+2R_2 \to R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$(A-2I)\vec{x} = 0$, $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$x_1 - x_3 = 0$: $x_1 = x_3$

$x_2 - x_3 = 0$: $x_2 = x_3$

$(x_3 = 1)$

$\therefore \vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\lambda = -1$

$$A - (-1)I = A + I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$(A+I)\vec{y} = \vec{0}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \end{bmatrix}$

$(A+I)\vec{y} = \vec{0}$ and $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\qquad \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow \quad y_1 = -y_2 - y_3$

$$\therefore \quad \vec{y} = \begin{bmatrix} -y_2 - y_3 \\ y_2 \\ y_3 \end{bmatrix} = y_2 \underbrace{\boxed{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}}_{\vec{v}} + y_3 \underbrace{\boxed{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}}_{\vec{w}}$$

$$\underbrace{\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\lambda = 2} \qquad \underbrace{\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \vec{w} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\lambda = -1}$$

Note: $\vec{x} \cdot \vec{v} = 0$
$x \cdot \vec{w} = 0$
$\vec{v} \cdot \vec{w} = 1 \neq 0$

Note: any linear combination of $\vec{v}, \vec{w}$ also an eigenvector for $\lambda = -1$.

$$\text{proj}_{\vec{v}} \vec{w} = \left(\frac{\vec{v} \cdot \vec{w}}{\vec{v} \cdot \vec{v}} \right) \vec{v} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix}$$

Let $\vec{u} = \vec{w} - \text{proj}_{\vec{v}} \vec{w} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$

use

$\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

Check: $\vec{x} \cdot \vec{v} = 0$
$\vec{x} \cdot \vec{u} = 0$
$\vec{u} \cdot \vec{v} = 0$

Get unit vectors

$\vec{a} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \vec{b} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{c} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ (orthonormal basis)

$$\vec{a} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad b = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \qquad c = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

(... basis)

(like $\hat{\imath}\ \hat{\jmath}\ \hat{k}$)

Let $$P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}.$$

(orthogonal matrix)

Check: $PP^T = I \implies P^T = P^{-1}$

and $$P^T A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Example 2

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$$A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & 3 \\ -1 & 2 & 2 \end{bmatrix}$$

$$\begin{array}{c} \quad + \quad\; - \quad\; + \end{array}$$

$$C_A(x) = \begin{vmatrix} x-1 & -2 & 0 \\ 3 & x-2 & -3 \\ 1 & -2 & x-2 \end{vmatrix} = (x-1)\begin{vmatrix} x-2 & -3 \\ -2 & x-2 \end{vmatrix} - (-2)\begin{vmatrix} 3 & -3 \\ 1 & x-2 \end{vmatrix}$$

$$= (x-1)\left((x-2)^2 - 6\right) + 2\left(3(x-2) + 3\right)$$

$$= (x-1)(x^2 - 4x + 4 - 6) + 6\underbrace{(x-2+1)}_{x-1}$$

$$= (x-1)\left[x^2 - 4x - 2 + 6\right]$$

$$= (x-1)(x^2 - 4x + 4) = (x-1)(x-2)^2$$

$\lambda = 1$

$\lambda = 2$ (mult. 2)

$\lambda = 1$

$$A - 1I = \begin{pmatrix} 1-1 & 2 & 0 \\ -3 & 2-1 & 3 \\ -1 & 2 & 2-1 \end{pmatrix} = \begin{pmatrix} 0 & 2 & 0 \\ -3 & 1 & 3 \\ -1 & 2 & 1 \end{pmatrix}$$

1

$$\longrightarrow \begin{pmatrix} 0 & 1 & 0 \\ -3 & 0 & 3 \\ -1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$(A - I)\vec{x} = \vec{0}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Longrightarrow \begin{array}{l} x_1 = x_3 \\ x_2 = 0 \end{array}$ take $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 2$

$$A - 2I = \begin{pmatrix} 1-2 & 2 & 0 \\ -3 & 2-2 & 3 \\ -1 & 2 & 2-2 \end{pmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ -3 & 0 & 3 \\ -1 & 2 & 0 \end{bmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$(A - 2I)\vec{y} = \vec{0}$, $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \Longrightarrow \begin{array}{l} y_1 = y_3 \\ 2y_2 = y_3 \end{array}$ $\quad \vec{y} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

$$y_2 = 1 \Rightarrow y_3 = 2 \Rightarrow y_1 = 2$$

$$A\vec{y} = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & 3 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \checkmark$$

– only one eigenvector!

$\therefore$ A can <u>not</u> be diagonalized.

What next? $\rightarrow$ generalized eigenvectors !!

Solve $(A-2I)^2 \vec{y} = \vec{0}$ (details....)

Get $\vec{x}, \vec{y}$, and $\vec{z}$ $\Leftarrow$ generalized eigenvector.

Set $P = \left[\vec{x} \mid \vec{y} \mid \vec{z} \right]$

$$P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

(Jordan canonical form)