Math 1410, Spring 2020

Matrix Transformations, Part 2

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Overview

Warm-Up

Assume T is a linear transformation.

1. Given $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\-2\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\5\end{bmatrix}$, find $T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$
2. Given $T(\vec{a}) = \begin{bmatrix}1\\0\\-3\end{bmatrix}$, $T(\vec{b}) = \begin{bmatrix}0\\-2\\5\end{bmatrix}$ and $T(\vec{c}) = \begin{bmatrix}1\\1\\1\end{bmatrix}$, find $T(2\vec{a} - 3\vec{b} + 5\vec{c})$.

Reminder: T is linear if $T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$ and $T(k\vec{x}) = kT(\vec{x})$ for any vectors $\vec{x}, \vec{y}$ and scalar k.

Is it linear?

For each map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$, decide if it's linear:

1. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x - 3y \\ x + 4y \end{bmatrix}$
2. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 4xy \\ 2x + y \end{bmatrix}$
3. $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 1 \\ x - 3y + 2 \end{bmatrix}$

Examples

1. If $T(\vec{x}) = A\vec{x}$ for $A = \begin{bmatrix} 2 & -3 \\ -1 & -2 \\ 5 & 4 \end{bmatrix}$, what are the domain and codomain of T?
2. For T as above, compute $T(\hat{\imath})$ and $T(\hat{\jmath})$
3. What if $A = \begin{bmatrix} 2 & -1 & 5 \\ 4 & -2 & 3 \end{bmatrix}$?
4. If $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x - 2y \\ -x + 4y + 5z \\ 7x - 2y - 6z \end{bmatrix}$, for what matrix is $T(\vec{x}) = A\vec{x}$?

More examples

1. Determine the matrix of $T : \mathbb{R}^3 \to \mathbb{R}^2$, given that

$$T(\hat{\imath}) = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, T(\hat{\jmath}) = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, T(\hat{k}) = \begin{bmatrix} -7 \\ 6 \end{bmatrix}.$$

2. Determine the matrix of $T : \mathbb{R}^2 \to \mathbb{R}^2$, given that

$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}, T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

When A is 2×2 we can visualize everything in terms of geometric vectors in the plane. We can use matrices to describe *transformations*, like stretches, rotations, and reflections. (But not translations.)

Example

Describe the effect of the transformation with matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ in terms of what it does to the *unit square* $(0 \leq x, y \leq 1)$

Transformation matrices

- *Stretches*: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}, \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = kI_2$. (This is just scalar multiplication.)
- *Reflections*: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- *Rotations*: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
- *Shears*: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Examples

Determine the matrix transformation that:

1. Stretches horizontally by a factor of 2, rotates by 90°, and then reflects across the x axis.
2. Reflects across the line $y = x$, stretches vertically be a factor of 3, then reflects across the y axis.

Column space

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. The *range* of T is the set of all $\vec{y} \in \mathbb{R}^m$ that are equal to $T(\vec{x})$ for some $\vec{x} \in \mathbb{R}^n$. In other words: if $T(\vec{x}) = A\vec{x}$, the range of T is the set of all $\vec{y}$ for which the system $A\vec{x} = \vec{y}$ is consistent.
Recall: if $A = \begin{bmatrix} A_1 & A_2 & \cdots A_n \end{bmatrix}$ (in terms of columns) then

$$A\vec{x} = x_1 A_1 + x_2 A_2 + \cdots + x_n A_n.$$

So the range of T is all linear combinations of the columns of A. (This is why range is also called *column space*.)

Examples

Determine the range of:

1. $S(\vec{x}) = A\vec{x}$, $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \\ -2 & -4 \end{bmatrix}$
2. $T(\vec{x}) = B\vec{x}$, $B = \begin{bmatrix} 1 & -3 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

Null space

The *null space* of a linear transformation $T(\vec{x}) = A\vec{x}$ is the set of all vectors $\vec{x}$ such that $T(\vec{x}) = \vec{0}$. In other words, it's the set of all solutions to the homogeneous system $A\vec{x} = \vec{0}$. Both null space and column space are examples of *subspaces*. Given a subspace, we often want to find a *basis* for it. This is a sort of "minimal generating set" of vectors. A basis for the null space is the set of basic solutions to $A\vec{x} = \vec{0}$.

Example

Determine the null space and column space of the transformation T with matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & -3 \\ -1 & -2 & 2 & 5 \\ 2 & 4 & -1 & -4 \end{bmatrix}.$$