Math 1410, Spring 2020

Gaussian Elimination

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Overview

Warm-up

What can you say about the set of solutions to each pair of equations below? (You might find it helpful to sketch the lines.)

1. $$\begin{aligned} x - 2y &= 5 \\ -4x + 8y &= -10 \end{aligned}$$

2. $$\begin{aligned} 4x + y &= 6 \\ 12x + 3y &= 18 \end{aligned}$$

3. $$\begin{aligned} 3x - y &= 4 \\ x + 2y &= 2 \end{aligned}$$

The augmented matrix

If we're going to be solving lots of linear systems (and we are), then it gets tedious quickly to keep track of all the variables. A "key idea": write only the coefficients, and keep track of variables by position. We organize everything in a rectangular array, called a **matrix**.

Example

System of equations:

$$\begin{array}{rcrcrcr} 2x & - & y & + & 4z & = & 7 \\ -5x & + & 7y & + & 8z & = & 11 \\ 3x & - & 2y & + & 81z & = & 12 \end{array}$$

Augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & -1 & 4 & 7 \\ -5 & 7 & 8 & 11 \\ 3 & -2 & 81 & 12 \end{array}\right].$$

From matrix to system

It's important to be able to convert in both directions:

Example

Write down the system (in variables x_1, x_2, x_3, x_4) corresponding to the augmented matrix

$$\left[\begin{array}{cccc|c} 3 & -2 & 1 & 4 & 6 \\ 1 & 0 & 2 & -7 & 12 \\ 0 & 3 & 4 & 0 & -38 \end{array}\right].$$

Elementary row operations

There are three elementary operations on the rows of a matrix, corresponding to the elementary operations on equations:

1. Swap any two rows: write $R_i \leftrightarrow R_j$
2. Multiply a row by a nonzero constant (rescale): write $cR_j \to R_j$
3. Add a multiple of one row to another: write $R_i + cR_j \to R_i$

Gaussian elimination

There is a standard algorithm for solving a system using row operations:

- By swapping rows and/or rescaling, get a 1 in row 1, column 1. (This is possible unless column 1 consists entirely of zeros.)
- By adding multiples of row 1 to the other rows, create zeros in all other entries of column 1.
- Move to row 2, column 2, and repeat (until you reach the last row or column).

Example

For the system below, write down the augmented matrix, and use Gaussian elimination to simplify.

$$\begin{array}{rcrcrcr} 2x & - & 4y & + & 2z & = & 8 \\ x & - & 3y & - & z & = & 5 \\ -x & + & 2y & + & z & = & 3 \end{array}$$

Then, solve the system, if possible.

Avoiding pitfalls

- Be careful of minus signs!
- Once first column looks like $\begin{bmatrix}1\\0\\0\\\vdots\\0\end{bmatrix}$, stop using R_1!
- To get next leading one, either divide, or use rows below.

RREF

How do we know when to stop the algorithm?

- Gaussian elimination works down and to the right, creating zeros and leading ones.
- Once each row "starts" with a leading one (first nonzero entry), and leading ones in lower rows are to the right of leading ones in higher rows, (only zeros below each leading one) you're in **row-echelon form**.
- Row-echelon form is not unique. From any REF you can probably solve by back-substitution.
- Eliminating non-zero entries *above* each leading one leads to **reduced row-echelon form**. This is unique, and as simplified as possible.

Examples

Which matrices are in REF? RREF? If neither, what's the next step?

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$