Math 1410, Spring 2020

Solving Systems of Equations

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Overview

Examples

Which matrices are in REF? RREF? If neither, what's the next step?

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad \left[\begin{array}{cccc|c} 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Gauss-Jordan Elimination

- *Forward steps:* perform Gaussian elimination to reach row-echelon form.
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    - You *could* solve by back substitution at this stage.
    - (If there's no solution, you should already be able to tell.)
    - Back substitution can be tricky if there are parameters involved.
- *Backward steps:* starting with right-most leading one, create zeros above, working up and to the left.

Example 1

Solve the system:

$$\begin{array}{rcrcrcr} 2x & - & y & + & 4z & = & 3 \\ x & - & y & + & 2z & = & -2 \\ -2x & + & 3y & - & z & = & 0 \end{array}$$

Example 2

Solve the system:

$$\begin{array}{rcrcrcr} x & - & 2y & + & 3z & = & 4 \\ 2x & - & 3y & + & 4z & = & -1 \\ x & - & 3y & + & 5z & = & 13 \end{array}$$

Example 3

Solve the system:

$$\begin{array}{rcrcr} 2x & - & 5y & = & 6 \\ x & - & 4y & = & 4 \\ -3x & + & 10y & = & -7 \end{array}$$

Example 4

Solve the system:

$$\begin{array}{rcrcrcrcr} x_1 & - & 3x_2 & & & + & 5x_4 & = & 4 \\ 2x_1 & - & 4x_2 & + & 6x_3 & - & 2x_4 & = & -9 \end{array}$$

Homogeneous systems

Definition:

A system of linear equations is homogeneous if all the constant terms (right-hand sides) are zero.

Example

$$\begin{array}{rcrcrcl} 3x & - & 5y & + & 2z & = & 0 \\ 2x & + & 2y & + & 4z & = & 0 \end{array}$$

For homogeneous systems, *existence* of solutions is not in question. (Why?) Instead, we're interested in whether solutions are *unique*.

Example

Solve the homogeneous system

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & - & 4x_4 & = & 0 \\ -2x_1 & + & 4x_2 & - & 3x_3 & - & 5x_4 & = & 0 \\ -x_1 & + & 2x_2 & + & x_3 & - & 13x_4 & = & 0 \end{array}$$

Vector solutions

It can be convenient to write our solutions in vector form. For later work with matrices, we use *column vectors*. Instead of giving solutions for $x_1, x_2, \ldots, x_n$ separately, we collect things into a vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

Vector solutions, continued

This can be notationally convenient. Instead of

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

write

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

or simply, $A\vec{x} = \vec{b}$.

Basic solutions

For a homogeneous system with parameters $t_1, t_2, \ldots, t_k$ in the vector solution, the **basic solutions** are obtained by setting one parameter equal to 1, and the others to 0. Notice that if $\vec{v}$ and $\vec{w}$ are both solutions to $A\vec{x} = \vec{0}$, then so is $s\vec{v} + t\vec{w}$ for any real numbers s, t.

Example

Find the basic solutions to

$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & - & 4x_4 & = & 0 \\ -2x_1 & + & 4x_2 & - & 3x_3 & - & 5x_4 & = & 0 \\ -x_1 & + & 2x_2 & + & x_3 & - & 13x_4 & = & 0 \end{array}$$