Math 1410, Spring 2020

Operations on Matrices

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Overview

Warm-Up

Determine the matrix transformation that:

1. Reflects across the y axis, stretches vertically by a factor of 3, and then rotates by 45°.
2. Reflects across the line $y = x$, stretches vertically be a factor of 3, then reflects across the y axis.

Transpose of a matrix

- The **transpose** of a matrix swaps its rows and columns.
- If $A = [a_{ij}]$ is an $m \times n$ matrix, then A^T is the $n \times m$ matrix whose (i, j) entry is a_{ji}.

Examples: on the board.

Why transpose?

Why should we care about transpose?

- Admittedly, it's one of several things we teach you to compute in Math 1410, and make you wait until later courses to really understand.
- Later, it gets related to *dual vectors* and *dual operators*, and is important in things like quantum mechanics.
- It gives us an easy way to turn columns into rows and vice versa: sometimes we write $\begin{bmatrix}1 & 2 & 3\end{bmatrix}^T$ instead of $\begin{bmatrix}1\\2\\3\end{bmatrix}$ because it fits better on the page.
- It also connects with the *dot product*: we can write $\vec{v}\cdot\vec{w} = (\vec{v})^T\vec{w}$.

Properties

In each case, assume that A and B have the right size for the operation to be defined. (True if both are $n \times n$ but things work for non-square matrices too.)

1. $(A+B)^T = A^T + B^T$
2. $(kA)^T = kA^T$
3. $(AB)^T = B^T A^T$
4. $(A^T)^T = A$
5. $(A^T)^{-1} = (A^{-1})^T$, if A is invertible.

Symmetric and antisymmetric matrices

An $n \times n$ matrix A is *symmetric* if

$$A^T = A$$

and *antisymmetric* if

$$A^T = -A.$$

What can we say about the entries of A in each case?

Constructing symmetric matrices

1. Show that $A + A^T$ is symmetric, and $A - A^T$ is antisymmetric, for any $n \times n$ matrix A
2. Show that AA^T and A^TA are symmetric for any matrix A.
3. Show that any square matrix A can be written as the sum of a symmetric and an antisymmetric matrix.

Trace of a matrix

The **trace** of a matrix is simply the sum of its diagonal entries. If $A = [a_{ij}]$ is an $n \times n$ matrix, then

$$\mathrm{tr}(A) = \sum_{k=1}^{n} a_{kk} = a_{11} + a_{22} + \cdots + a_{nn}.$$

(If A is $m \times n$ with $m \neq n$, sum to whichever of m, n is smaller.)
Examples? On the board!

Properties of the trace

1. $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$
2. $\mathrm{tr}(kA) = k\,\mathrm{tr}(A)$
3. $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ (as long as both products are defined)
4. $\mathrm{tr}(A^T) = \mathrm{tr}(A)$

Note: on the set of all $m \times n$ matrices, the pairing

$$\langle A, B\rangle = \mathrm{tr}(B^T A)$$

defines a sort of "dot product".

Introduction

The **determinant** is a function that assigns a *number* to any square matrix A. We denote this number by $\det A$ or $|A|$. We'll define $\det A$ *recursively*, starting with 2×2 matrices, and then showing how to reduce the determinant of a larger matrix to smaller ones. There is a general formula, but it's........ complicated.

The 2×2 case

You've already learned how to compute these, back when we did cross products! Given $A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$, $\det A = ad - bc$:

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

Note:

1. If $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ is parallel to $\vec{v} \begin{bmatrix} c \\ d \end{bmatrix}$, then $\det A = 0$.
2. Otherwise, $\det A$ calculates (up to sign) the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

The 3×3 case

This is also not a big leap from cross products:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}.$$

You might recall that this is the same as the *scalar triple product* $\vec{a} \cdot (\vec{b} \times \vec{c})$. Example: compute the determinant of $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & 1 \\ -1 & 0 & 5 \end{bmatrix}$.

Minors and Cofactors

Given an $n \times n$ matrix $A = [a_{ij}]$,

- The (i, j) *minor* of A is the $(n-1) \times (n-1)$ matrix M_{ij} obtained by deleting row i and column j of A
- The (i, j) *cofactor* of A is the number C_{ij} defined by

$$C_{ij} = (-1)^{i+j} \det M_{ij}.$$

Note: $(-1)^{i+j}$ equals $+1$ if $i + j$ is even, and -1 if $i + j$ is odd. (Now do some examples, Sean)

Next steps

- In general, we define $\det A = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$ (cofactor expansion along first row). (Each cofactor C_{1j} is a determiant one smaller than A)
- Then we prove (OK, state assertively) that we can actually expand any row or column of A.
- Then we'll observe that determinants of *triangular* matrices are really easy.
- Finally, we'll see what happens to the determinant if we use row operations to get a matrix into triangular form.