Math 1410, Spring 2020

Introduction to Matrix Algebra

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Overview

1. Introduction
2. Linear Systems as Matrix-Vector Equations
3. Matrix algebra

Warm-Up

Find the general solution to the homogeneous ystem

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & + & x_3 & - & 4x_4 & = & 0 \\ -2x_1 & + & 4x_2 & - & 3x_3 & + & 5x_4 & = & 0 \\ -x_1 & + & 2x_2 & - & 3x_3 & - & 2x_4 & = & 0 \end{array}.$$

Vector solutions

It can be convenient to write our solutions in vector form. For later work with matrices, we use *column vectors*. Instead of giving solutions for $x_1, x_2, \ldots, x_n$ separately, we collect things into a vector $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. For homogeneous systems this lets us easily determine the **basic solutions**. If our general solution has parameters $t_1, t_2, \ldots, t_k$, the basic solutions are obtained by setting one parameter equal to 1, and the others to 0.

Vector solutions, continued

This can be notationally convenient. A system

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array}$$

has three main pieces:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

Matrix-vector product

We want to write our system in the form $A\vec{x} = \vec{b}$. How do we define the product $A\vec{x}$?

- As a linear combination:

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} x_{1n} \\ x_{2n} \\ \vdots \\ x_{mn} \end{bmatrix}.$$

- Using row-times-column "dot products":

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n \end{bmatrix}.$$

- Both produce the same result! (Let's try an example or two.)

Properites of the product

Let A be an $m \times n$ matrix, and let $\vec{x}$ be an $k \times 1$ column vector.

1. The product $A\vec{x}$ is only defined if $k = n$
2. The result is an $m \times 1$ column vector.
3. For $n \times 1$ vectors $\vec{v}, \vec{w}$, $A(\vec{v} + \vec{w}) = A\vec{v} + A\vec{w}$.
4. For any $n \times 1$ vector $\vec{v}$ and scalar c, $A(c\vec{v}) = c(A\vec{v})$.

Matrix-Vector Form

With our new notation, we can write a system of equations as $A\vec{x} = \vec{b}$.

- This is notationally convenient: it's an easy way to refer to a general system of equations.
- It's meant to remind you of the case of one equation with one variable: $ax = b$.
- It can be used to quickly establish facts about systems.

Examples

Example

Verify that $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$ is a solution to

$$\begin{bmatrix} 3 & -1 & 2 \\ -2 & 2 & -5 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 9 \\ 5 \end{bmatrix}.$$

Example

Show that if $\vec{v}$ is a solution to the homogeneous system $A\vec{x} = \vec{0}$, and $\vec{w}$ is a solution to the non-homogeneous system $A\vec{x} = \vec{b}$, then $\vec{v} + \vec{w}$ is also a solution to $A\vec{x} = \vec{b}$.

Rank and structure of solutions

The **rank** of a matrix is the number of leading ones in reduced row-echelon form. Given a system $A\vec{x} = \vec{b}$ where A is $m \times n$, note:

- There are m equations.
- There are n variables.
- If $\operatorname{rank}(A) = r$, then the general solution to $A\vec{x} = \vec{0}$ has $k = n - r$ parameters.
- If $\operatorname{rank}\left(A \mid \vec{b} \right) = \operatorname{rank}(A)$, the system $A\vec{x} = \vec{b}$ is consistent, and the general solution can be written as $\vec{x} = \vec{x}_p + \vec{x}_h$, where $\vec{x}_p$ is a *particular* solution to $A\vec{x} = \vec{b}$, and $\vec{x}_h$ is the general solution to the homogeneous system $A\vec{x} = \vec{0}$.
- If $\operatorname{rank}\left(A \mid \vec{b} \right) > \operatorname{rank}(A)$, then the system is inconsistent.

Examples

Determine the rank of A, and solve $A\vec{x} = \vec{b}$, where

1. $A = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & -2 & 2 & 0 \\ 2 & -3 & 7 & -2 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

Matrices

By now, we've seen plenty of matrices. A **matrix** is just a rectangular array of numbers, arranged into rows and columns. If a matrix has m rows and n columns, we say it has **size** (or *shape*) $m \times n$. Some examples of matrices:

$$A = \begin{bmatrix} 2 & -5 & 0 \\ \ln(42) & 1410 & 2020 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Particular entries in a matrix are referenced by row and column. We write a_{ij} for the (i, j)-entry of a matrix A.

Addition and scalar multiplication

This works exactly like you'd expect. Given matrices $A = [a_{ij}], B = [b_{ij}]$ of the same size, define

$$A + B = [a_{ij} + b_{ij}].$$

For any scalar c, define $cA = [ca_{ij}]$. Aside: what do we mean by "$=$"?

Properties

Let A, B, and C be matrices of the same size. Then:

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A + 0 = A$ (here 0 is the *zero matrix*)
4. $c(A + B) = cA + cB$
5. $(c + d)A = cA + dA$
6. $c(dA) = (cd)A$