Math 1410, Spring 2020

Determinants

Sean Fitzpatrick

Overview

Warm-Up

Compute the determinant of the given matrices:

1. $A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$
2. $B = \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 4 \\ 0 & 1 & -5 \end{bmatrix}$

Minors and Cofactors

Given an $n \times n$ matrix $A = [a_{ij}]$,

- The (i, j) *minor* of A is the $(n-1) \times (n-1)$ matrix M_{ij} obtained by deleting row i and column j of A
- The (i, j) *cofactor* of A is the number C_{ij} defined by

$$C_{ij} = (-1)^{i+j} \det M_{ij}.$$

Note: $(-1)^{i+j}$ equals $+1$ if $i + j$ is even, and -1 if $i + j$ is odd. (Now do some examples, Sean)

The Determinant ($n \times n$)

Definition:

Let $A = [a_{ij}]$ be an $n \times n$ matrix. The *determinant* of A is the number $\det A$ given by

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n},$$

where C_{ij} refers to the (i, j) cofactor of A.

The sum above is called a *cofactor expansion*. Note that if A is 4×4 or larger, this definition is recursive: each C_{ij} is a 3×3 determinant, which must be computed using cofactor expansion in terms of 2×2 determinants.

Example

Compute the determinant of

$$A = \begin{bmatrix} 2 & 1 & 0 & 4 \\ 0 & -1 & 2 & 3 \\ 3 & 0 & 5 & -2 \\ 1 & 2 & -1 & 0 \end{bmatrix}.$$

Laplace expansion theorem

Theorem:

The determinant can be computed using cofactor expansion along any row.

In fact, it turns out $\det A = \det A^T$ for any $n \times n$ matrix A, so we can also do cofactor expansion along any *column*.

Example

Compute the determinant of $A = \begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 5 & 0 & 0 \\ 3 & 0 & 1 & -2 \\ 4 & -4 & 0 & 1 \end{bmatrix}$

Triangular matrices

Definition:

A matrix $A = [a_{ij}]$ is called *upper-triangular* if $a_{ij} = 0$ whenever $i > j$, and *lower-triangular* if $a_{ij} = 0$ whenever $i < j$. A matrix $D = [d_{ij}]$ is called *diagonal* if $d_{ij} = 0$ whenever $i \neq j$.

An upper triangular matrix has all zeros below the main diagonal. A lower triangular matrix has all zeros above the main diagonal. A diagonal matrix has zeros both above and below the main diagonal. It also counts as triangular.

Examples

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- This is a slide containing very little, other than to let us know that Sean is about to write some triangular matrices on the board, and then compute their determinants.
- OK, this slide also contains a theorem: if A is a triangular matrix, then $\det A$ is given by the product of the diagonal entries of A:

$$\det A = a_{11}a_{22}\cdots a_{nn}.$$

Effect of row operations

Moral of the story so far:

- Determinants are generally hard to compute. (Well, not so much hard as annoying.)
- Except if the matrix is triangular. Then determinants are easy.
- We know how to put a matrix into triangular form. (Row echelon form is triangular!)
- Looks like we'd better figure out what row operations do to a determinant!

Theorem:

1. If B is obtained from A using the row operation $R_i \leftrightarrow R_j$, then $\det B = -\det A$.
2. If B is obtained from A using the row operation $kR_i \to R_i$, then $\det B = k\det A$.
3. If B is obtained from A using the row operation $R_i + kR_j \to R_i$, then $\det B = \det A$.

Examples

Compute the determinant of:

1. $A = \begin{bmatrix} 2 & 6 & -4 \\ -1 & 2 & 3 \\ 2 & 5 & 1 \end{bmatrix}$
2. $B = \begin{bmatrix} 6 & -3 & 1 \\ -2 & 1 & 4 \\ 2 & 5 & -3 \end{bmatrix}$ (Is there a more efficient option than finding triangular form?)
3. $C = \begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 3 & -2 & 1 \\ -1 & 3 & 0 & -4 \\ -2 & 1 & -3 & 5 \end{bmatrix}$

Properties of Determinants

Theorem:

Let A and B be $n \times n$ matrices. Then:

1. $\det(AB) = \det(A)\det(B)$
2. $\det(A^T) = \det(A)$
3. $\det(kA) = k^n \det(A)$

Theorem:

A matrix A is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$