Math 1410, Spring 2020

Matrix Algebra

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Overview

Warm-Up

Let $A = \begin{bmatrix} 3 & -1 & 2 \\ -1 & 4 & 0 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}, \vec{w} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$. Confirm that $A(2\vec{v} + \vec{w}) = 2(A\vec{v}) + A\vec{w}$.

Basic matrix products

- An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns.
- A *row vector* is a $1 \times k$ matrix $R = \begin{bmatrix} a_1 & a_2 & \cdots & a_k \end{bmatrix}$ consisting of a single row.
- A *column vector* is a $k \times 1$ matrix $C = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_k \end{bmatrix}$ consisting of a single column.
- The product RC is a dot product: $RC = a_1b_1 + a_2b_2 + \cdots + a_kb_k$. ($R$ and C must have the same number of entries)
- To compute $A\vec{x}$ where A is $m \times n$ and $\vec{x}$ is $n \times 1$, multiply each row in A by $\vec{x}$.

Addition and scalar multiplication

This works exactly like you'd expect. Given matrices $A = [a_{ij}], B = [b_{ij}]$ of the same size, define

$$A + B = [a_{ij} + b_{ij}].$$

For any scalar c, define $cA = [ca_{ij}]$. Aside: what do we mean by "="?

Examples

Let $A = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} -7 & 2 \\ 5 & -1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 5 & -4 \\ 3 & -2 & 1 \end{bmatrix}$. Compute: $A + C, 3B, 2A - 3C$. What can you say about $A + B$, and $C + A$?

Properties

Let A, B, and C be matrices of the same size. Then:

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $A + 0 = A$ (here 0 is the *zero matrix*)
4. $c(A + B) = cA + cB$
5. $(c + d)A = cA + dA$
6. $c(dA) = (cd)A$

Examples

Given $A = \begin{bmatrix} 2 & -3 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & -5 \\ 4 & -3 \end{bmatrix}, C = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}$, compute

$$(2A - 3B) - 4(A - B + C) + 3(A + C).$$

With A, B, C as above, find X such that

$$4(X - A) + B = 2C.$$

Recap

So far we know:

1. How to multiply a row by a column.
2. To multiply A by $\vec{x}$, multiply each row of A by $\vec{x}$. Arrange results in a column vector.

Example

Calculate $A\vec{x}$, where $A = \begin{bmatrix} 1 & -4 \\ 3 & 2 \\ 5 & 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

General products

We now want to compute AB, where A and B are matrices.
We still do "row times column", using rows from A, columns from B.
Need each row in A to have same length as each column in B: if A is $m \times n$, B is $n \times p$.
The product AB is size $m \times p$. Its (i, j)-entry is the (dot) product of row i from A and column j from B.
Note: write $B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix}$ where $\vec{b}_j$ is column j of B. Then

$$AB = A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix}.$$

Examples

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & -1 \\ -1 & 2 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 4 \\ -3 & 2 \\ 2 & -2 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Compute, if possible:

$$AB, AC, A^2, BC, CB, CA, AD, DA.$$

Compare $A(BC)$ with $(AB)C$

Properties

Definition:

I_n denotes the $n \times n$ *identity matrix*. The *diagonal* entries of I_n (when $i = j$) all equal 1; all other entries are 0.

Theorem:

Assuming each product below is defined,

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $A(kB) = (kA)B = k(AB)$ for any scalar k
5. $I_m A = A$ and $AI_n = A$

Matrix inverses

For real numbers a and b, if $a \neq 0$ and $ax = b$, we know $x = \frac{b}{a}$. If A is a matrix, $\vec{b}$ is a vector, and $A\vec{x} = \vec{b}$, can we similarly solve for $\vec{x}$? Short answer: no. Longer answer: sometimes. Sort of.

Definition:

A square ($n \times n$) matrix A is *invertible* if $AB = I_n = BA$ for some B. We call B the *inverse* of A, and write $B = A^{-1}$.

Example

For $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, we saw that $AD = I_2$ and $DA = I_2$. So $D = A^{-1}$. Suppose $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$. How can we use D to solve for $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$?