Math 1410, Spring 2020

Dot and Cross Products

January 24, 2020

Overview

Warm-up

Given vectors $\vec{v} = \langle 2, -1, 3 \rangle$ and $\vec{w} = \langle -4, 5, 1 \rangle$, find:

1. $\vec{v} + \vec{w}$
2. $3\vec{v} - 2\vec{w}$
3. $\|\vec{v}\|$

Scalar multiplication

Recall: Given a real number c and a vector $\vec{v} = \langle a, b \rangle$,

$$c\vec{v} = c \langle a, b \rangle = \langle ca, cb \rangle .$$

The story in $\mathbb{R}^3$ is similar:

$$c \langle x, y, z \rangle = \langle cx, cy, cz \rangle .$$

Some observations:

1. For any vector $\vec{v}$, $0\vec{v} = \vec{0}$ and $(-1)\vec{v} = -\vec{v}$.
2. We have $\vec{v} + \vec{v} = 2\vec{v}$.
3. In general, $a\vec{v} + b\vec{v} = (a + b)\vec{v}$.
4. Also, $c(\vec{v} + \vec{w}) = c\vec{v} + c\vec{w}$.

Scalar multiplication, geometrically

- $2\vec{v} = \vec{v} + \vec{v}$, so $2\vec{v}$ is in the same direction as $\vec{v}$, but twice as long.
- In general, we have:

Theorem:

For any vector $\vec{v}$ and scalar (number) c,

$$\|c\vec{v}\| = |c| \|\vec{v}\| .$$

Parallel vectors, unit vectors

- **Definition: Parallel vectors.**

  We say that two vectors $\vec{v}$ and $\vec{w}$ are **parallel** if $\vec{w} = c\vec{v}$ for some scalar c.

- **Definition: Unit vector.**

  A vector $\vec{u}$ is a **unit vector** if $\|\vec{u}\| = 1$.

- Unit vectors are useful when we care about direction, but not magnitude.
- Given $\vec{v} = \langle 2, 3 \rangle$, what is a unit vector in the direction of $\vec{v}$?

Standard unit vectors

In $\mathbb{R}^2$:

$$\hat{\imath} = \langle 1, 0 \rangle \,, \quad \hat{\jmath} = \langle 0, 1 \rangle \,.$$

In $\mathbb{R}^3$:

$$\hat{\imath} = \langle 1, 0, 0 \rangle \,, \quad \hat{\jmath} = \langle 0, 1, 0 \rangle \,, \quad \hat{k} = \langle 0, 0, 1 \rangle \,.$$

Using the standard unit vectors

Write the vector $\vec{v} = \langle 4, -7, 6 \rangle$ in terms of the vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$.

Dot Products

The *dot product* provides the algebra – geometry bridge.

Definition:

Let $\vec{v} = \langle v_1, v_2 \rangle, \vec{w} = \langle w_1, w_2 \rangle$ be vectors in $\mathbb{R}^2$. The **dot product** $\vec{v} \cdot \vec{w}$ is given by

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2.$$

For $\vec{v} = \langle v_1, v_2, v_3 \rangle, \vec{w} = \langle w_1, w_2, w_3 \rangle$ in $\mathbb{R}^3$, we similarly have

$$\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3.$$

Examples

For $\vec{v} = \langle 3, -4 \rangle$, $\vec{w} = \langle 6, 2 \rangle$:

- Compute $\vec{v} \cdot \vec{w}$
- Compute $\vec{v} \cdot (3\vec{w})$
- Compute $3(\vec{v} \cdot \vec{w})$

For $\vec{u} = \langle 2, -1, 3 \rangle$, $\vec{v} = \langle -3, -1, 4 \rangle$, $\vec{w} = \langle 0, 1, -5 \rangle$:

- Compute $\vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
- Compute $\vec{u} \cdot (\vec{v} + \vec{w})$

Properties of the dot product

Theorem:

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, and let c be a scalar. Then:

1. $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
2. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
3. $\vec{u} \cdot (c\vec{v}) = (c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$
4. $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$
5. $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos(\theta)$, where θ is the angle between $\vec{v}$ and $\vec{w}$.

Orthogonal vectors

The dot product lets us compute angles between vectors. Example:

$$\vec{v} = \langle 2, -1 \rangle, \vec{w} = \langle 3, 2 \rangle.$$

Most useful for us: when $\theta = \pi/2$.

Definition:

We say that two vectors $\vec{v}, \vec{w}$ are **orthogonal** if $\vec{v} \cdot \vec{w} = 0$.

Example

Decide if the triangle with vertices $P = (1, 0, 2)$, $Q = (3, -1, 0)$, $R = (4, 3, -1)$ is a right-angled triangle.

Orthogonal projection

This is probably the most important application of the dot product in Math 1410. To give it proper attention, we'll hold it over to Thursday's class.

Cross products

- Defined for vectors in $\mathbb{R}^3$ only.
- Produces a *vector* rather than a scalar.
- Cross product $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.
- Definition: if $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$,

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle .$$

Example

Let $\vec{u} = \langle 3, -1, 2 \rangle, \vec{v} = \langle 0, 2, -1 \rangle, \vec{w} = \langle -2, 0, 4 \rangle$. Find:

- $\vec{u} \times \vec{v}$
- $\vec{v} \times \vec{w}$
- $\vec{w} \times \vec{v}$

Areas and angles

If the angle between $\vec{v}$ and $\vec{w}$ is θ,

$$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \, \|\vec{w}\| \sin\theta.$$

The direction of $\vec{v} \times \vec{w}$ given by "right-hand rule". Useful to note: if $\vec{v}$ and $\vec{w}$ form 2 of 4 sides of a parallelogram, that parallelogram has area $A = \|\vec{v}\| \, \|\vec{w}\| \sin\theta$.

Examples

Let $P = (0, 2, -1), Q = (3, 1, -2), R = (4, -2, 0), S = (7, -3, -1)$. Verify that the quadrilateral with these vertices is a parallelogram, and find its area. What about the triangle ΔPQR?