Math 1410, Spring 2020

Lines in three dimensions

January 24, 2020

Overview

Warm-up

For the vectors $\vec{u} = \langle 3, -14 \rangle$, $\vec{v} = \langle -2, -2, 5 \rangle$, $\vec{w} = \langle 0, 4, -3 \rangle$, find:

1. $\vec{u} \cdot (2\vec{v} - \vec{w})$
2. $(2\vec{u}) \cdot \vec{v} - \vec{u} \cdot \vec{w}$
3. A value of c such that $\vec{u}$ is orthogonal to $\vec{v} + c\vec{w}$

Orthogonal vectors

Recall: for vectors $\vec{v}, \vec{w}$, $\vec{v} \cdot \vec{w} = \|\vec{v}\| \, \|\vec{w}\| \cos\theta$, where θ is the angle between $\vec{v}$ and $\vec{w}$. Most useful for us: when $\theta = \pi/2$.

Definition:

We say that two vectors $\vec{v}, \vec{w}$ are **orthogonal** if $\vec{v} \cdot \vec{w} = 0$.

Cross products

- Defined for vectors in $\mathbb{R}^3$ only.
- Produces a *vector* rather than a scalar.
- Cross product $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.
- Definition: if $\vec{v} = \langle v_1, v_2, v_3 \rangle$, $\vec{w} = \langle w_1, w_2, w_3 \rangle$,

$$\vec{v} \times \vec{w} = \langle v_2 w_3 - v_3 w_2, v_3 w_1 - v_1 w_3, v_1 w_2 - v_2 w_1 \rangle .$$

(Easier to remember using "determinant trick".)

Example

Let $\vec{u} = \langle 3, -1, 2 \rangle, \vec{v} = \langle 0, 2, -1 \rangle, \vec{w} = \langle -2, 0, 4 \rangle$. Find:

- $\vec{u} \times \vec{v}$
- $\vec{v} \times \vec{w}$
- $\vec{w} \times \vec{v}$

Properties

1. $\vec{v} \times \vec{w} = -\vec{w} \times \vec{v}$
2. $\vec{v} \times (c\vec{w}) = (c\vec{v}) \times \vec{w} = c(\vec{v} \times \vec{w})$
3. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
4. $\vec{v} \times \vec{w}$ is orthogonal to both $\vec{v}$ and $\vec{w}$.

Areas and angles

If the angle between $\vec{v}$ and $\vec{w}$ is θ,

$$\|\vec{v} \times \vec{w}\| = \|\vec{v}\| \ \|\vec{w}\| \sin\theta.$$

The direction of $\vec{v} \times \vec{w}$ given by "right-hand rule". Useful to note: if $\vec{v}$ and $\vec{w}$ form 2 of 4 sides of a parallelogram, that parallelogram has area $A = \|\vec{v}\| \ \|\vec{w}\| \sin\theta$.

Examples

Let $P = (0, 2, -1), Q = (3, 1, -2), R = (4, -2, 0), S = (7, -3, -1)$. Verify that the quadrilateral with these vertices is a parallelogram, and find its area. What about the triangle ΔPQR?

Lines the plane

We all know lines in the plane: $y = mx + b$ gives a line with slope m passing through $(0, b)$. Does slope make sense in $\mathbb{R}^3$?

Example

Describe the line through the points $(1, -2)$ and $(3, 4)$:

1. Using a "slope-intercept" equation
2. Using vectors

Lines in space

- In $\mathbb{R}^3$, to specify a line we need a *point* (on the line), and a *direction* (vector).
- Suppose $P_0 = (x_0, y_0, z_0)$ and $P = (x, y, z)$ are two points on a line in the direction of a vector $\vec{v} = \langle a, b, c \rangle$. What can we say about the vector $\overrightarrow{P_0P}$?

Vector and parametric equations

Points on a line in space are given in terms of a **parameter** (usually t — we can think of motion in a straight line, with t as time).

- The *vector* equation of a line through $P_0 = (x_0, y_0, z_0)$ in the direction of $\vec{v} = \langle a, b, c\rangle$ is

$$\langle x, y, z, \rangle = \langle x_0, y_0, z_0\rangle + t\,\langle a, b, c\rangle\,,$$

  or $\vec{x} = \vec{x}_0 + t\vec{v}$, for short.
- Sometimes see $\vec{r}$ or $\vec{r}(t)$ instead of $\vec{x}$.
- Equating coefficients in the vector equation gives the *parametric equations*:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct. \end{aligned}$$

Examples

Find the vector equations of the lines:

1. Through the point $P_0 = (2, -5, 1)$ in the direction of $\vec{v} = \langle 6, -7, 3 \rangle$
2. Through the points $P = (4, 5, -3)$ and $Q = (-1, 8, 2)$.
3. Through the point $P_0 = (4, 5, -9)$ and parallel to the line

$$\vec{r}(t) = \langle 3 - 4t, -6 + 8t, 2 - 7t \rangle .$$

Comparing lines: intersecting, parallel, skew

- Two lines in $\mathbb{R}^3$ are **parallel** if their direction vectors are parallel.
- Not all paralell lines intersect: some are **skew**.
- Checking for intersection leads to a system of equations. (Be sure to use a different parameter for each line.)

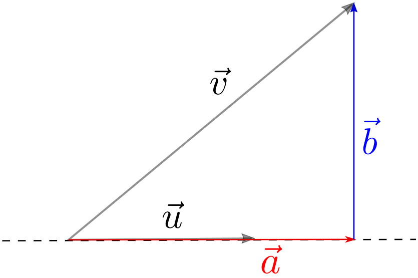
Example

Determine if the following pairs of lines are parallel, skew, or if they intersect.

1. $\vec{r}_1(s) = \langle 3, -2, 4\rangle + s\,\langle 4, -2, 6\rangle$ and $\vec{r}_2(t) = \langle 2, 2, 2\rangle + t\,\langle -6, 3, -9\rangle$
2. $\vec{r}_1(s) = \langle 3, 1, 1\rangle + s\,\langle 2, -1, 3\rangle$ and $\vec{r}_2(t) = \langle 1, 0, -1\rangle + t\,\langle 1, 2, 1\rangle$
3. $\vec{r}_1(s) = \langle 0, 1, 2\rangle + s\,\langle 4, -2, 1\rangle$ and $\vec{r}_2(t) = \langle -2, -3, 7\rangle + t\,\langle 3, 1, -2\rangle$

Orthogonal projection

Given vectors $\vec{u}$ (not equal to $\vec{0}$) and $\vec{v}$, often useful to write $\vec{v}$ as the sum of a vector *parallel* to $\vec{u}$, and a vector *orthogonal* to $\vec{u}$.



Vector $\vec{a}$ called the **projection** of $\vec{v}$ onto $\vec{u}$. Notation and formula:

$$\text{proj}_{\vec{u}}\,\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u}.$$

Distance

Example

1. Find the distance from the point $P = (3, 1, 2)$ to the line

$$\langle x, y, z \rangle = \langle 0, 2, -1 \rangle + t \langle 2, -1, 2 \rangle .$$

2. Find the distance between the parallel lines

$$\vec{r}_1(t) = \langle 2, -1, 3 \rangle + t \langle 1, 2, 3 \rangle , \text{ and } \vec{r}_2(t) = \langle -4, 1, 3 \rangle + t \langle 2, 4, 6 \rangle .$$