Math 1410, Spring 2020

Determinant Properties and Applications — Pandemic Lockdown Style

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Overview

1. Recap

2. Properties of Determinants

3. The adjugate formula for the inverse

Warm-Up

Compute the determinant of the given matrices, possibly after doing a row operation:

1. $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & -3 & 5 \\ 0 & -1 & -2 \end{bmatrix}$
2. $B = \begin{bmatrix} 3 & 0 & -2 & 1 \\ -2 & 1 & 1 & 3 \\ 0 & -1 & 2 & -3 \\ 4 & 0 & 1 & 0 \end{bmatrix}$

Effect of row operations

Theorem:

1. If B is obtained from A using the row operation $R_i \leftrightarrow R_j$, then $\det B = -\det A$.
2. If B is obtained from A using the row operation $kR_i \rightarrow R_i$, then $\det B = k \det A$.
3. If B is obtained from A using the row operation $R_i + kR_j \rightarrow R_i$, then $\det B = \det A$.

Note: these effects are most easily observed in elementary matrices!

Examples

1. Suppose B is obtained from A using the folloing row operations:
   1. $\frac{1}{4}R_1 \to R_1$
   2. $R_2 - 4R_1 \to R_2$
   3. $R_2 \leftrightarrow R_3$
   4. $R_3 + 3R_2 \to R_3$

   If $\det B = -7$, what is $\det A$?
2. If A is a 4×4 matrix and $\det A = -3$, what is the value of $\det(2A)$?

Properties of Determinants

Theorem:

Let A and B be $n \times n$ matrices. Then:

1. $\det(AB) = \det(A)\det(B)$
2. $\det(A^T) = \det(A)$
3. $\det(kA) = k^n \det(A)$

Theorem:

A matrix A is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

Examples

Given that $\det A = 3$ and $\det B = -2$, what is the value of:

1. $\det(A^2B^3)$
2. $\det(B^{-1}AB)$
3. $\det(2AB^{-1})$

More examples

What can you say about $\det A$ if:

1. $A^2 = A$
2. $A^4 = I$
3. $PA = P$, where P is invertible.

The cofactor matrix

Recall: given an $n \times n$ matrix A, the (i,j) cofactor is the number $C_{ij} = (-1)^{i+j} \det M_{ij}$, where M_{ij} is the (i,j) minor. The *matrix of cofactors* of A is the matrix $\mathrm{cof}(A)$ whose (i,j) entry is C_{ij}. Example: find $\mathrm{cof}(A)$ if $A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -1 & 0 \end{bmatrix}$.

The adjugate matrix

Definition:

The *adjugate* of an $n \times n$ matrix A is given by $\operatorname{adj}(A) = \operatorname{cof}(A)^T$.

Theorem:

For any $n \times n$ matrix A,

$$A \cdot \operatorname{adj}(A) = \det(A) I_n.$$

Examples

Use the formula $A^{-1} = \dfrac{1}{|A|}\operatorname{adj}(A)$ to compute the inverse of:

1. $A = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$
2. $A = \begin{bmatrix} 1 & 0 & x \\ 0 & -x & 2 \\ x & 0 & 3 \end{bmatrix}$.

Cramer's Rule

Suppose we have a system of n equations in n unknowns, written as $A\vec{x} = \vec{b}$. If $\det A = 0$, then A is not invertible, and this system has either no solution, or infinitely many solutions. If $\det A \neq 0$, then

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{|A|}\,\mathrm{adj}(A)\vec{b}.$$

Result: if A_i denotes the matrix obtained by replacing column i of A by $\vec{b}$, then

$$x_i = \frac{\det A_i}{\det A},$$

for $i = 1, 2, \ldots, n$. (Theoretically and historically interesting, but not very practical.)

Example

Use Cramer's rule to solve the system:

$$(\cos\theta)x - (\sin\theta)y = 4$$
$$(\sin\theta)x + (\cos\theta)y = W,$$

where θ is an angle and W is some unknown (but presumably very important) number.