Math 1410, Spring 2020

Determinant Properties and Applications — Pandemic Lockdown Style

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Overview

1. Recap
2. Properties of Determinants
3. The adjugate formula for the inverse

Warm-Up

$\det(B) = b_{12}C_{12} + b_{22}C_{22} + b_{32}C_{32} + b_{42}C_{42}$

$ \;0 \qquad\quad 1 \qquad\quad -1 \qquad\quad 0$

Compute the determinant of the given matrices, possibly after doing a row operation:

$2(-1)^{2+1}\begin{vmatrix} 4 & 0 \\ -1 & -2 \end{vmatrix} + (-3)(-1)^{2+2}\begin{vmatrix} 1 & 0 \\ 0 & -2 \end{vmatrix} + 5(-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 0 & -1 \end{vmatrix}$

$R_2 - 2R_1 \to R_2$

1. $A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & -3 & 5 \\ 0 & -1 & -2 \end{bmatrix}$

$\det(A) = \begin{vmatrix} 1 & 4 & 0 \\ 0 & -11 & 5 \\ 0 & -1 & -2 \end{vmatrix}$

$= 1(+1)\begin{vmatrix} -11 & 5 \\ -1 & -2 \end{vmatrix}$

$= 22 - (-5) = 27$

2. $B = \begin{bmatrix} 3 & 0 & -2 & 1 \\ -2 & 1 & 1 & 3 \\ 0 & -1 & 2 & -3 \\ 4 & 0 & 1 & 0 \end{bmatrix}$ $(1,2)$

$R_3 + R_2 \to R_3$ $\quad \det(B) = \begin{vmatrix} 3 & 0 & -2 & 1 \\ -2 & 1 & 1 & 3 \\ -2 & 0 & 3 & 0 \\ 4 & 0 & 1 & 0 \end{vmatrix}$

$\det(M_{22})$

$= 1\begin{vmatrix} 3 & -2 & 1 \\ -2 & 3 & 0 \\ 4 & 1 & 0 \end{vmatrix}$

$= 1 \cdot 1\begin{vmatrix} -2 & 3 \\ 4 & 1 \end{vmatrix}$

$= -2 - 12 = -14.$

$C_{ij} = (-1)^{i+j}\det(M_{ij})$

minor

Effect of row operations

$R_1 + R_2 \to R_2$ ✓
$-R_1 + R_2 \to R_2$ ✓
$R_1 - R_2 \to R_2$ ✗

Theorem:

1. If B is obtained from A using the row operation $R_i \leftrightarrow R_j$, then $\det B = -\det A$.
2. If B is obtained from A using the row operation $kR_i \to R_i$, then $\det B = k \det A$.
3. If B is obtained from A using the row operation $R_i + kR_j \to R_i$, then $\det B = \det A$.

$kR_j - R_i \to R_i$ is not elementary!!

Note: these effects are most easily observed in elementary matrices!

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\det(I)=1} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = B$$

$\det(B) = -1$

$R_3 \to 5R_3$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$R_1 - 4R_3 \to R_1$

$\begin{bmatrix} 1 & 0 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Examples

$B = E_4 E_3 E_2 E_1 A$

$|B| = (1)(-1)(1)(1/4)|A|$

1. Suppose B is obtained from A using the folloing row operations:
   1. $\frac{1}{4}R_1 \to R_1$
   2. $R_2 - 4R_1 \to R_2$
   3. $R_2 \leftrightarrow R_3$
   4. $R_3 + 3R_2 \to R_3$

   If $\det B = -7$, what is $\det A$?
2. If A is a 4×4 matrix and $\det A = -3$, what is the value of $\det(2A)$?

$A \xrightarrow{①} B_1 \xrightarrow{②} B_2 \xrightarrow{③} B_3 \xrightarrow{④} B$

$\det B_1 = \det B_2$

$= \frac{1}{4}\det(A)$

$\det B = \det B_3 = -\det B_2$

$= -\frac{1}{4}\det(A)$

$-\frac{1}{4}\det(A) = \det(B) = -7 \implies \det(A) = 28$

$A \longrightarrow 2A$

$\times 2 \quad 2R_1 \to R_1$

$\times 2 \quad 2R_2 \to R_2$

$\times 2 \quad 2R_3 \to R_3$

$\times 2 \quad 2R_4 \to R_4$

$\det(2A) = 2^4(-3)$

$= 16(-3)$

$= -48$

Properties of Determinants

Theorem:

Let A and B be $n \times n$ matrices. Then:

1. $\det(AB) = \det(A)\det(B)$
2. $\det(A^T) = \det(A)$
3. $\det(kA) = k^n \det(A)$

$|AB| = |A||B|$

– can't say anything about $\det(A+B)$

Theorem:

A matrix A is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then

$$\det(A^{-1}) = \frac{1}{\det(A)}.$$

$AA^{-1} = I$

$|AA^{-1}| = |I|$

$|A||A^{-1}| = 1$

$\therefore |A^{-1}| = 1/|A|$

Examples

A, B are 3×3.

Given that $\det A = 3$ and $\det B = -2$, what is the value of:

1. $\det(A^2B^3)$ $= \det(A \cdot A \cdot B \cdot B \cdot B)$
$= |A||A||B||B||B|$
$= |A|^2|B|^3 = 3^2(-2)^3 = -72.$
2. $\det(B^{-1}AB)$
3. $\det(2AB^{-1})$

$= \det(B^{-1})\det(A)\det(B)$

$= \frac{1}{\cancel{\det(B)}}\det(A)\cancel{\det(B)} = 3.$

$\det(2AB^{-1}) = \det(2A)\det(B^{-1})$
$= 2^3|A| \cdot \frac{1}{|B|} = \frac{8(3)}{-2} = -12.$

More examples

What can you say about $\det A$ if:

1. $A^2 = A$
2. $A^4 = I$
3. $PA = P$, where P is invertible. $\quad |P| \neq 0$

$$|A|^4 = 1 \qquad x^4 = 1 \Rightarrow x = \pm 1$$

$$A^2 = A$$

$$A \cdot A = A$$

$$|A| \cdot |A| = |A|$$

or $\quad |A|^2 - |A| = 0$

or $\quad |A|(|A| - 1) = 0$

$$|A| = 0 \text{ or } 1$$

$$|PA| = |P|$$

$$\frac{|P||A|}{|P|} = \frac{|P|}{|P|}$$

$$\therefore |A| = 1.$$

The cofactor matrix (Determinants and A^{-1})

Recall: given an $n \times n$ matrix A, the (i,j) cofactor is the number $C_{ij} = (-1)^{i+j} \det M_{ij}$, where M_{ij} is the (i,j) minor. The *matrix of cofactors* of A is the matrix $\mathrm{cof}(A)$ whose (i,j) entry is C_{ij}. Example: find

$$\mathrm{cof}(A) \text{ if } A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -1 & 0 \end{bmatrix}.$$

$$C_{11} = (-1)^{1+1} \begin{vmatrix} 4 & -2 \\ -1 & 0 \end{vmatrix} = -2$$

$$C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & -2 \\ 1 & 0 \end{vmatrix} = -2$$

$$C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 4 \\ 1 & -1 \end{vmatrix} = -4$$

$$C_{21} = - \begin{vmatrix} -1 & 3 \\ -1 & 0 \end{vmatrix} = -3$$

$$C_{22} = + \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} = -3$$

$$C_{23} = - \begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} = 1$$

$$C_{31} = + \begin{vmatrix} -1 & 3 \\ 4 & -2 \end{vmatrix} = -10$$

$$C_{32} = - \begin{vmatrix} 2 & 3 \\ 0 & -2 \end{vmatrix} = 4$$

$$C_{33} = + \begin{vmatrix} 2 & -1 \\ 0 & 4 \end{vmatrix} = 8$$

$$\mathrm{cof}(A) = \begin{bmatrix} -2 & -2 & -4 \\ -3 & -3 & 1 \\ -10 & 4 & 8 \end{bmatrix}$$

The adjugate matrix

Definition:

The *adjugate* of an $n \times n$ matrix A is given by $\operatorname{adj}(A) = \operatorname{cof}(A)^T$.

Theorem:

For any $n \times n$ matrix A,

$$A \cdot \operatorname{adj}(A) = \det(A) I_n.$$

$$A \cdot \left(\frac{1}{|A|}\operatorname{adj}(A)\right) = I$$

$$\therefore\ A^{-1} = \frac{1}{|A|}\operatorname{adj}(A)$$

$$\begin{bmatrix} -2 & -2 & -4 \\ -3 & -3 & 1 \\ -10 & 4 & 8 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 & -10 \\ -2 & -3 & 4 \\ -4 & 1 & 8 \end{bmatrix} = \operatorname{adj}(A)$$

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 4 & -2 \\ 1 & -1 & 0 \end{bmatrix} \qquad A \cdot \operatorname{adj}(A) = \begin{bmatrix} -14 & 0 & 0 \\ 0 & -14 & 0 \\ 0 & 0 & -14 \end{bmatrix} = -14 I_3$$

$|A|$

Examples

Use the formula $A^{-1} = \dfrac{1}{|A|}\,\mathrm{adj}(A)$ to compute the inverse of:

1. $A = \begin{bmatrix} 2 & 1 & -3 \\ 3 & 0 & 2 \\ 0 & 1 & 4 \end{bmatrix}$ $\longleftarrow$ Exercise
2. $A = \begin{bmatrix} 1 & 0 & x \\ 0 & -x & 2 \\ x & 0 & 3 \end{bmatrix}$.

$$A^{-1} = \frac{1}{x^3-3x}\begin{pmatrix} -3x & 0 & x^2 \\ 2x & 3-x^2 & -2 \\ x^2 & 0 & -x \end{pmatrix}$$

$$x \neq 0, \pm\sqrt{3}$$

$$\mathrm{cof}(A) = \begin{pmatrix} -3x & +2x & x^2 \\ 0 & 3-x^2 & 0 \\ x^2 & -2 & -x \end{pmatrix}$$

$$|A| = -x \begin{vmatrix} 1 & x \\ x & 3 \end{vmatrix}$$

$$= -x(3-x^2)$$
$$= x^3-3x = x(x^2-3)$$

Cramer's Rule

Suppose we have a system of n equations in n unknowns, written as $A\vec{x} = \vec{b}$. If $\det A = 0$, then A is not invertible, and this system has either no solution, or infinitely many solutions. If $\det A \neq 0$, then

$$\vec{x} = A^{-1}\vec{b} = \frac{1}{|A|}\,\mathrm{adj}(A)\vec{b}.$$

Result: if A_i denotes the matrix obtained by replacing column i of A by $\vec{b}$, then

$$x_i = \frac{\det A_i}{\det A}$$

for $i = 1, 2, \ldots, n$. (Theoretically and historically interesting, but not very practical.)

Example

Use Cramer's rule to solve the system:

$$\begin{aligned}(\cos\theta)x - (\sin\theta)y &= 4\\ (\sin\theta)x + (\cos\theta)y &= W,\end{aligned}$$

where θ is an angle and W is some unknown (but presumably very important) number.

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$|A| = \cos^2\theta + \sin^2\theta = 1$$

$$A_1 = \begin{bmatrix} 4 & -\sin\theta \\ W & \cos\theta \end{bmatrix}$$

$$|A_1| = 4\cos\theta + W\sin\theta = x$$

$$A_2 = \begin{bmatrix} \cos\theta & 4 \\ \sin\theta & W \end{bmatrix} = W\cos\theta - 4\sin\theta = y.$$