Math 1410, Spring 2020

Lines and Planes

January 24, 2020

Overview

Warm-up

1. Compute the cross product of $\vec{v} = \langle 3, -1, 4\rangle$ and $\vec{w} = \langle 2, 0, -1\rangle$.
2. Find a vector equation for the line through $P = (1, 2, 3)$ and $Q = (3, -1, 4)$

Vector and parametric equations of lines

Points on a line in space are given in terms of a **parameter** (usually t — we can think of motion in a straight line, with t as time).

- The *vector* equation of a line through $P_0 = (x_0, y_0, z_0)$ in the direction of $\vec{v} = \langle a, b, c \rangle$ is

  $$\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + t \langle a, b, c \rangle ,$$

  or $\vec{x} = \vec{x}_0 + t\vec{v}$, for short.
- Sometimes see $\vec{r}$ or $\vec{r}(t)$ instead of $\vec{x}$.
- Equating coefficients in the vector equation gives the *parametric equations*:

  $$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct. \end{aligned}$$

Finding equations

Find the vector equations of the lines:

1. Through the point $P_0 = (2, -5, 1)$ in the direction of $\vec{v} = \langle 6, -7, 3 \rangle$
2. Through the points $P = (4, 5, -3)$ and $Q = (-1, 8, 2)$.
3. Through the point $P_0 = (4, 5, -9)$ and parallel to the line

$$\vec{r}(t) = \langle 3 - 4t, -6 + 8t, 2 - 7t \rangle .$$

Comparing lines: intersecting, parallel, skew

- Two lines in $\mathbb{R}^3$ are **parallel** if their direction vectors are parallel.
- Not all parallel lines intersect: some are **skew**.
- Checking for intersection leads to a system of equations. (Use a different parameter for each line.)

Example

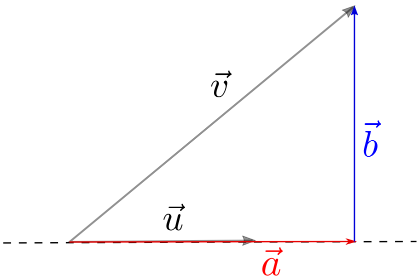
Determine if the following pairs of lines are parallel, skew, or if they intersect.

1. $$\vec{r}_1(s) = \langle 3, -2, 4 \rangle + s\langle 4, -2, 6 \rangle$$
   $$\vec{r}_2(t) = \langle 2, 2, 2 \rangle + t\langle -6, 3, -9 \rangle$$

2. $$\vec{r}_1(s) = \langle 3, 1, 1 \rangle + s\langle 2, -1, 3 \rangle$$
   $$\vec{r}_2(t) = \langle 1, 0, -1 \rangle + t\langle 1, 2, 1 \rangle$$

Orthogonal projection

Given vectors $\vec{u}$ (not equal to $\vec{0}$) and $\vec{v}$, often useful to write $\vec{v}$ as the sum of a vector *parallel* to $\vec{u}$, and a vector *orthogonal* to $\vec{u}$.



Vector $\vec{a}$ called the **projection** of $\vec{v}$ onto $\vec{u}$. Notation and formula:

$$\mathrm{proj}_{\vec{u}}\,\vec{v} = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u}.$$

Orthogonal decomposition

Given $\vec{v} = \langle 3, -1 \rangle$ and $\vec{w} = \langle 2, 5 \rangle$, find vectors $\vec{a}$ and $\vec{b}$ such that:

1. $\vec{a}$ is parallel to $\vec{v}$
2. $\vec{b}$ is orthogonal to $\vec{v}$
3. $\vec{a} + \vec{b} = \vec{w}$

Distance

Example

1. Find the distance from the point $P = (3, 1, 2)$ to the line

$$\langle x, y, z \rangle = \langle 0, 2, -1 \rangle + t \langle 2, -1, 2 \rangle .$$

2. Find the distance between the parallel lines

$$\vec{r}_1(t) = \langle 2, -1, 3 \rangle + t \langle 1, 2, 3 \rangle , \text{ and } \vec{r}_2(t) = \langle -4, 1, 3 \rangle + t \langle 2, 4, 6 \rangle .$$

Equations of planes

Two ways to describe a plane:

1. A point, and *two* vectors parallel to the plane.
2. A point, and *one* vector perpendicular to the plane.

Second option is simpler. Suppose $\vec{n} = \langle a, b, c \rangle$ is perpendicular to the plane. Suppose also $P_0 = (x_0, y_0, z_0)$ and $P = (x, y, z)$ are in the plane. Then $\overrightarrow{P_0P}$ is parallel to the plane, so $\vec{n} \cdot \overrightarrow{P_0P} = 0$.

Finding equations

Find the equation of the plane:

1. Through $P_0 = (2, -1, 3)$ and perpendicular to $\vec{n} = \langle 5, -3, 4 \rangle$
2. Perpendicular to the line $\vec{r}(t) = \langle 3 + 2t, -5 + 3t, -2 - 9t \rangle$, and containing the origin.
3. Containing the points $P = (1, 0, 4), Q = (2, -1, 3), R = (5, 2, 4)$
4. Containing the lines

$$\vec{r}_1(s) = \langle 0, 1, 2 \rangle + s \langle 4, -2, 1 \rangle$$
$$\vec{r}_2(t) = \langle -2, -3, 7 \rangle + t \langle 3, 1, -2 \rangle$$

(We found that these intersect at $P_0 = (4, -1, 3)$.)