Math 1410, Spring 2020

Planes and Distances

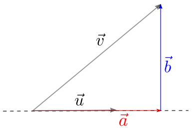
January 24, 2020

Overview

Warm-up

Given $\vec{u} = \langle 2, 0, -1 \rangle$ and $\vec{v} = \langle 1, 3, -1 \rangle$, find vectors $\vec{a}$ and $\vec{b}$ such that:

- $\vec{a}$ is parallel to $\vec{u}$
- $\vec{b}$ is orthgonal to $\vec{u}$
- $\vec{a} + \vec{b} = \vec{v}$



Vector $\vec{a}$ is called the **projection** of $\vec{v}$ onto $\vec{u}$. Notation and formula:

$$\text{proj}_{\vec{u}}\, \vec{v} = \left(\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|^2} \right) \vec{u}.$$

Distance

Example

1. Find the distance from the point $P = (3, 1, 2)$ to the line

$$\langle x, y, z\rangle = \langle 0, 2, -1\rangle + t\,\langle 2, -1, 2\rangle\,.$$

   What is the point on the line that is closest to P?

2. Find the distance between the parallel lines

$$\vec{r}_1(t) = \langle 2, -1, 3\rangle + t\,\langle 1, 2, 3\rangle\,, \text{ and } \vec{r}_2(t) = \langle -4, 1, 3\rangle + t\,\langle 2, 4, 6\rangle\,.$$

Planes in $\mathbb{R}^3$

Two ways to describe a plane:

1. A point, and *two* vectors parallel to the plane.
2. A point, and *one* vector perpendicular to the plane.

Second option is simpler.

- Suppose $\vec{n} = \langle a, b, c \rangle$ is perpendicular to the plane.
- Suppose also $P_0 = (x_0, y_0, z_0)$ and $P = (x, y, z)$ are in the plane.
- Then $\overrightarrow{P_0P}$ is parallel to the plane, so $\vec{n} \cdot \overrightarrow{P_0P} = 0$.

Finding equations

Find the equation of the plane:

1. Through $P_0 = (2, -1, 3)$ and perpendicular to $\vec{n} = \langle 5, -3, 4 \rangle$
2. Perpendicular to the line $\vec{r}(t) = \langle 3 + 2t, -5 + 3t, -2 - 9t \rangle$, and containing the origin.
3. Containing the points $P = (1, 0, 4), Q = (2, -1, 3), R = (5, 2, 4)$
4. Containing the lines

$$\vec{r}_1(s) = \langle 0, 1, 2 \rangle + s \langle 4, -2, 1 \rangle$$
$$\vec{r}_2(t) = \langle -2, -3, 7 \rangle + t \langle 3, 1, -2 \rangle$$

(We found that these intersect at $P_0 = (4, -1, 3)$ on Tuesday.)

Intersection: plane and a line

Suppose we have:

1. A plane $ax + by + cz = d$
2. A line $\langle x, y, z \rangle = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

How do we find what point (if any) they have in common?

Example

Find the point of intersection of the plane

$$2x - 3y + 4z = 6$$

and the line

$$\langle x, y, z \rangle = \langle 2, -1, 0 \rangle + t \langle 1, 1, -2 \rangle .$$

Intersection: two planes

Like parallel lines in $\mathbb{R}^2$, parallel planes in $\mathbb{R}^3$ do not interesect, but any non-parallel planes do. What does the intersection look like? How do we find it?

Example

Find the intersection of the planes $x - 2y + 3z = 2$ and $3x - y - 4z = 8$.

Two methods:

1. Using cross products
2. Solving a system

Distance: point to plane

- Yes, there's a formula in the book.
- Don't use it.
- The formula lets you get an answer without understanding what's going on. (That is not a good thing.)

Setup is similar to point–to–line distance, but we project onto a *normal vector*, not a direction vector. Given a point P and a plane $ax + by + cz = d$:

- Choose a point P_0 on the plane.
- Form the vector $\overrightarrow{P_0P}$. This goes from the plane to the point, but probably not at a right angle.
- Project $\overrightarrow{P_0P}$ onto the normal vector $\vec{n} = \langle a, b, c \rangle$. Result goes from plane to point, and is as short as possible.

Example

Example

Find the distance from the point $P = (1, 4, 1)$ to the plane $2x - y + 3z = 4$. Also find the point Q on the plane that is closest to P.

Two possible methods for solving the problem:

1. Using projection
2. Using a normal line

Distance: parallel planes

Example

Find the distance between the planes $2x - 3y + z = 4$ and $2x - 3y + z = 10$.

Distance: skew lines

Example

Find the distance between the skew lines

$$\vec{r}_1(s) = \langle 3, 1, 1 \rangle + s \langle 2, -1, 3 \rangle$$
$$\vec{r}_2(t) = \langle 1, 0, -1 \rangle + t \langle 1, 2, 1 \rangle .$$

Bonus opportunity: find the two points (one on each line) that have this minimum distance.