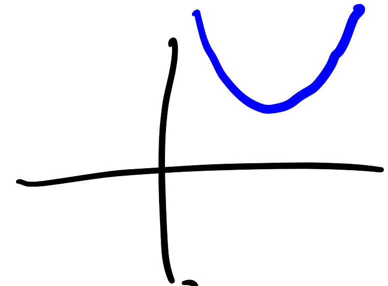
Complex Numbers

Math 1410 Linear Algebra

March 24, 2020

Warm-up

Use the quadratic formula to solve the equation



$$(x-2)^2+1=0$$

$$x^2 - 4x + 5 = 0.$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \Rightarrow x = \frac{4 \pm \sqrt{16-20}}{2}$$

(for $ax^2+bx+c=0$)

$a=1$, $b=-4$ $c=5$

$$= \frac{2 \pm \sqrt{-4}}{2}$$

$$= \frac{2 \pm 2i}{2} = 2 \pm i$$

Introduce i, where $i^2=-1$

$\Rightarrow$ $(2i)^2 = 4i^2 = -4$

The complex number system

Ref. Visual Complex Analysis by Tristan Needham

We define the set of complex numbers, denoted $\mathbb{C}$, as

real numbers

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R}\},$$

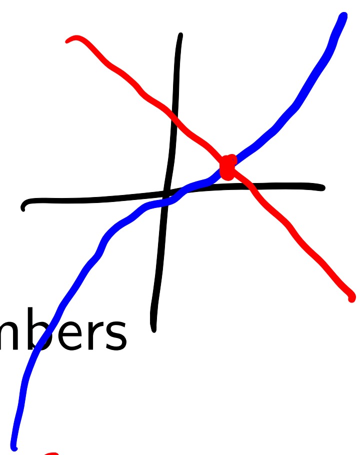
where i denotes a (non-real) number with the property that $i^2 = -1$. $\left(i = \sqrt{-1} \right)$

- Complex numbers date back to 16th-century Italy, and Cardano's *Ars Magna* (1545).
- Used (relucantly) by Bombelli to solve equations in 1572.
- Largely ignored as nonsense for 250 years. (Some dabbling by Euler around 1770.)
- Acceptance follows geometric interpretation by Gauss, Argand, and others at the end of the 18th century.
- Most development of the subject (by Cauchy, Riemann, et al) took place between 1814 and 1851.

Bombelli and the cubic

- It's easy to assume that complex numbers arose out of the need to solve quadratic equations like $x^2 + 1 = 0$ or $x^2 - 4x + 5 = 0$.
- This is historically false: geometrically no solutions were expected. (The equation $y = x^2 - 4x + 5 = (x-2)^2 + 1$ describes a parabola that lies above the x axis.)
- First compelling reason was the cubic equation $x^3 = 3px + 2q$.
- Cubic formula due to Cardano:

$$x = \sqrt[3]{q + \sqrt{q^2 - p^3}} + \sqrt[3]{q - \sqrt{q^2 - p^3}}$$

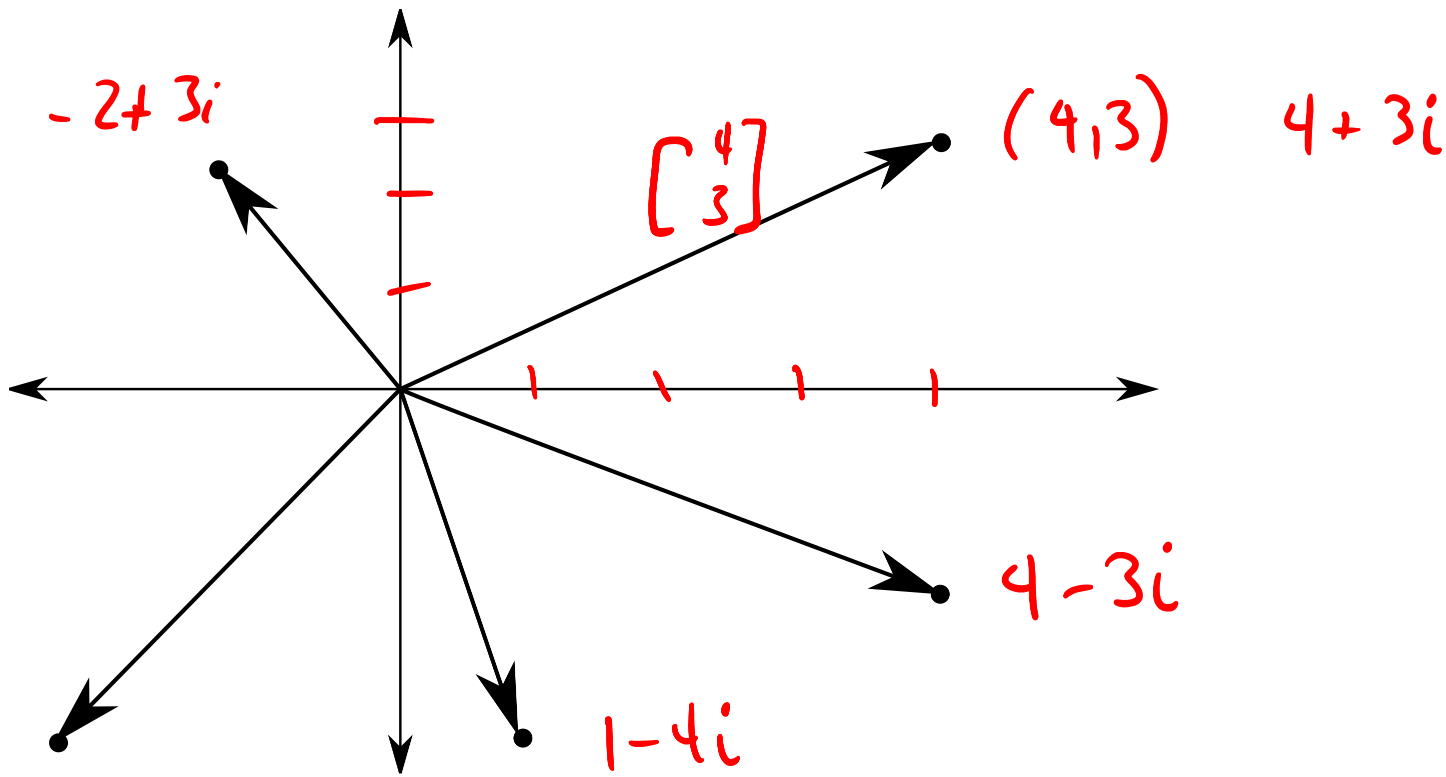


- Result is a real number even if there are negative numbers under the square roots.

$x = \sqrt[3]{q + in} + \sqrt[3]{q - in}$ negative if $p^3 > q^2$

The Argand Plane

Geometrically, we identify $z = x + iy \in \mathbb{C}$ with $(x, y) \in \mathbb{R}^2$. This visualization is usually called the Argand plane or Gauss plane, after the mathematicians who introducded this point of view.



Addition of complex numbers

$z = x + iy$ real part imaginary part

Addition is the same as the addition of geometric vectors in $\mathbb{R}^2$:
If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then we define

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$$

$$\vec{v}_1 + \vec{v}_2 = \langle a_1, b_1 \rangle + \langle a_2, b_2 \rangle = \langle a_1 + a_2, b_1 + b_2 \rangle$$

Examples:

$$(2+3i) + (-4+7i) = (2-4) + (3i+7i)$$
$$= -2 + 10i$$

$$((-7+2i) + (6-5i)) + (2+7i)$$
$$= ((-7+6) + i(2-5)) + (2+7i)$$
$$= (-1-3i) + (2+7i)$$
$$= (-1+2) + (-3i+7i)$$
$$= 1 + 4i$$

$$-7+6+2$$
$$+ 2i - 5i + 7i$$
$$= 1 + 4i$$

Properties of addition

Addition in $\mathbb{C}$ follows the same rules as addition in $\mathbb{R}$ (or $\mathbb{R}^2$):

- $z_1 + z_2 = z_2 + z_1$ for all $z_1, z_2 \in \mathbb{C}$.
- $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$
- $0 + z = z + 0 = z$ for all $z \in \mathbb{C}$, where $0 = 0 + i0$.
- Given $z = x + iy$, if we define $-z = -x - iy$, then $z + (-z) = -z + z = 0$.

Same as $\mathbb{R}$

$z_1 = x_1 + iy_1$
$z_2 = x_2 + iy_2$

$$\Rightarrow z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$
$$= (x_2 + x_1) + i(y_2 + y_1) = z_2 + z_1$$

Eg: Solve $az + b = c$, $a = 2 + i$, $b = 3 - 2i$, $c = -4 + 3i$

$$\Rightarrow az = c - b$$
$$\therefore \frac{1}{a}(az) = \frac{1}{a}(c - b) \Rightarrow z = \frac{c - b}{a}$$

Multiplication of complex numbers

A big difference between $\mathbb{C}$ and $\mathbb{R}^2$ (algebraically) is that we can multiply complex numbers.
Given $z = x + iy$ and $w = u + iv$, zw is computed using "FOIL", where we remember that $i^2 = -1$:

$$zw = (x + iy)(u + iv) = xu + ixv + iyu + i^2 yv = (xu - yv) + i(xv + yu)$$

Examples:

$$= -2(2+3i)(2-3i) = -2(2^2+3^2)$$

$$(2+3i)(-4+6i) = 2(-4) + 2(6i) - 4(3i) + (3i)(6i)$$

$$= -8 + 12i - 12i + 18(-1) = -26$$

$$-4+6i = -2(2-3i)$$

$$\boxed{(-1+4i)(2-3i) = -2+3i+8i \underbrace{-12i^2}_{+12} = 10+11i.}$$

$z = 2+3i$

$\bar{z} = 2-3i$ (complex conjugate)

$z = a+ib$

$$z\bar{z} = (a+ib)(a-ib) = a^2 - b^2(-1) = a^2+b^2$$

Multiplicative inverses

Given $z \in \mathbb{C}$ with $z \neq 0$, can we find a complex number z^{-1} (or $1/z$) such that $zz^{-1} = 1$? $AA^{-1} = I$

Say $z = x + iy$ and $w = u + iv$ satisfy $zw = 1$. Then

$$zw = (xu - yv) + i(xv + yu) = 1 = 1 + i0,$$

which gives a system of equations in u and v:

$$xu - yv = 1 \quad \text{and} \quad xv + yu = 0.$$

Solving (Cramer's Rule, anyone?) gives $u = \dfrac{x}{x^2 + y^2}$ and $v = \dfrac{-y}{x^2 + y^2}$, which suggests:

$$\frac{\bar{z}}{\bar{z}} \cdot \frac{1}{z} = \frac{1}{x + iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2} = \frac{x}{|z|^2} - i\frac{y}{|z|^2} = \frac{\bar{z}}{|z|^2}$$

Properties of complex multiplication

The previous slide tells us every nonzero complex number has a multiplicative inverse. We can check that the following properties (the same ones $\mathbb{R}$ has!) all hold:

- $z_1 z_2 = z_2 z_1$ for all $z_1, z_2 \in \mathbb{C}$.
- $z_1(z_2 z_3) = (z_1 z_2) z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$.
- $1z = z1 = z$ for all $z \in \mathbb{C}$, where $1 = 1 + i0$.
- For all $z \neq 0$, there exists $z^{-1} \in \mathbb{C}$ such that $zz^{-1} = z^{-1}z = 1$
- $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$ for all $z_1, z_2, z_3 \in \mathbb{C}$

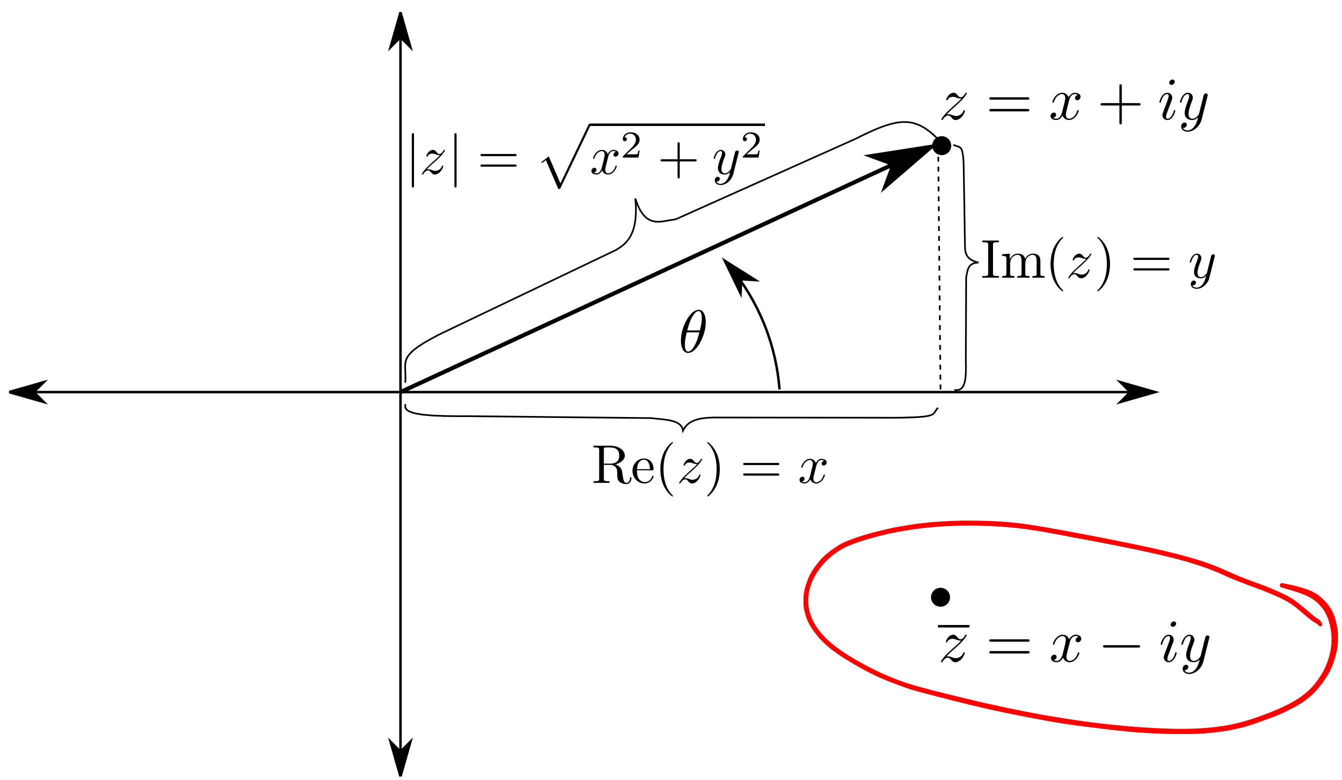
Same as for $\mathbb{R}$.

Elements of complex numbers

For any $z = x + iy \in \mathbb{C}$, we define the following:

- The real part of z, $\operatorname{Re} z = x$.
- The imaginary part of z, $\operatorname{Im} z = y$.
- The complex conjugate of z, $\bar{z} = x - iy$.
- The modulus of z, $|z| = \sqrt{x^2 + y^2}$.
- The argument of z, $\arg z = \theta$, where $\tan \theta = \frac{y}{z}$.

$|z|^2 = x^2 + y^2 = z\bar{z}$



Division in $\mathbb{C}$

We saw above that for every non-zero $z \in \mathbb{C}$, we can define $z^{-1} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$ such that $zz^{-1} = 1$.
In principle, this allows us to define division, but the result is hard to remember. Instead, we note the following:

Theorem

For all $z \in \mathbb{C}$, $z\bar{z} = |z|^2$.

$$z = \frac{|z|^2}{\bar z} \Rightarrow \frac{1}{z} = \frac{\bar z}{|z|^2}$$

Proof. Let $z = x + iy$, so $\bar z = x - iy$

$$\therefore z\bar z = (x+iy)(x-iy) = x^2 - i\cancel{x}y + i\cancel{x}y - y^2 \overset{-1}{i^2}$$
$$= x^2 + y^2$$

$\square$

How does it help? $|z|^2 = x^2 + y^2$ is a real number, and we know how to divide by real numbers. Dividing can be done by "multiplying by the conjugate".

Examples

1. Given $z = 3+5i$. Find $\frac{1}{z}$.

$$\frac{1}{3+5i} = \frac{1}{(3+5i)} \cdot \frac{(3-5i)}{(3-5i)} = \frac{3-5i}{3^2+5^2}$$

$$= \frac{3}{34} - \frac{5}{34} i$$

2. Given $z = -2+4i$, $w = 3-4i$, compute:

$$\frac{z}{w} = \frac{(-2+4i)}{(3-4i)} \cdot \frac{(3+4i)}{(3+4i)} = \frac{-6-8i+12i \overset{i^2}{-} 16}{\underset{3^2+4^2}{3^2+(-4)^2}}$$

$$= -\frac{22}{25} + \frac{4}{25} i.$$

$w = 3-4i = u+iv$

$u=3$, $v=-4$

$|w| = \sqrt{u^2+v^2} = \sqrt{3^2+(-4)^2}$

$(3-4i)(3+4i) = 3\cdot 3 + 12i - 12i + (-4)(4)i^2$

A matrix model for $\mathbb{C}$

If you find the idea of defining a number i such that $i^2 = -1$, weird consider the following:

Let V denote the set of all 2×2 matrices. Let $\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in V$,

and recall that this matrix satisfies $A\mathbf{1} = \mathbf{1}A = A$ for all $A \in V$.
Now, let $U \subseteq V$ denote the subset

$$U = \left\{ \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \middle| \, x, y \in \mathbb{R} \right\}.$$

Notice that if $A \in U$, then $A = x\mathbf{1} + y\mathbf{i}$, where $\mathbf{i} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

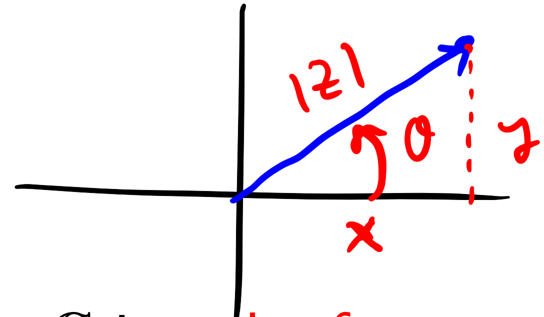
$$i^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -\mathbb{1}$$

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} c & d \\ -d & c \end{bmatrix} = \begin{bmatrix} ac - bd & ad + bc \\ ad + bc & ac - bd \end{bmatrix}$$

The polar form of a complex number

Given $z = x + iy$, we have $|z| = \sqrt{x^2 + y^2}$, and if $\theta = \arg z = \tan^{-1}(y/x)$, then basic trigonometry tells us

$$x = |z| \cos\theta$$
$$y = |z| \sin\theta$$



Thus, if we let $r = |z|$, we can write any $z \in \mathbb{C}$ in polar form as

$$z = r\cos\theta + ir\sin\theta. = r(\cos\theta + i\sin\theta)$$

Note: For the above to be well-defined, we have to define a "branch" of the argument: there are infinitely many values of θ that work. We'll require $\theta \in (-\pi, \pi]$. (Some texts take $\theta \in [0, 2\pi)$.)

Euler's Formula

Euler's identity is one of the most remarkable formulas in mathematics: we define the complex exponential $e^{i\theta}$ by

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

$z = re^{i\theta}$

This is usually taken as a definition, although there are several motivations for it. In particular, using the angle addition trigonometric identities, we find that

$$\begin{aligned} e^{i(\alpha+\beta)} &= \cos(\alpha+\beta) + i\sin(\alpha+\beta) \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ &= (\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) \\ &= \underline{e^{i\alpha}}\,\underline{e^{i\beta}} \end{aligned}$$

Another famous result, called de Moivre's theorem, asserts that for all natural numbers n,

$$(e^{i\theta})^n = (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta = e^{in\theta}.$$

Polar form, again

If we introduce polar coordinates $r = \sqrt{x^2 + y^2}$ and $\theta \in (-\pi, \pi]$ such that $\tan\theta = y/x$, we can write

$$z = |z|\cos\theta + i|z|\sin\theta = re^{i\theta},$$

with the help of Euler's theorem. This form of a complex number can be very convenient. For one thing, we will see that it makes finding roots of complex numbers much easier. For another, it gives us a geometric interpretation of complex multiplication.

$z = 2e^{i\pi/4}$ $\quad w = 3e^{i(\pi/3)}$

$zw = 6e^{i\,7\pi/12}$

The Fundamental Theorem of Algebra

Complex numbers don't just allow us to solve quadratic or cubic equations involving real numbers. In fact, we have the following remarkable theorem:

Theorem

Let $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ *be any polynomial with complex coefficients* $a_0, a_1, \ldots, a_n \in \mathbb{C}$*,* $n \geq 1$*. Then* p *has a root.*

Consequence: every polynomial – even with complex coefficients – can be completely factored over $\mathbb{C}$.