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Matrix Inverses

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Overview

Warm-Up

$$A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 0 & -1 \\ -1 & 2 & 3 \end{bmatrix} \qquad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 4 \\ -3 & 2 \\ 2 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Compute the products:

$$AB, BA, BC, CB, AD, DA, I_2A, I_2B, BI_3, I_3C.$$

Compare $A(BC)$ with $(AB)C$

Properties

Theorem:

Assuming each product below is defined,

1. $A(BC) = (AB)C$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $A(kB) = (kA)B = k(AB)$ for any scalar k
5. $I_m A = A$ and $AI_n = A$

Matrix inverses

For real numbers a and b, if $a \neq 0$ and $ax = b$, we know $x = \frac{b}{a}$. If A is a matrix, $\vec{b}$ is a vector, and $A\vec{x} = \vec{b}$, can we similarly solve for $\vec{x}$? Short answer: no. Longer answer: sometimes. Sort of.

Definition:

A square ($n \times n$) matrix A is *invertible* if $AB = I_n = BA$ for some B. We call B the *inverse* of A, and write $B = A^{-1}$.

Example

For $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$, we saw that $AD = I_2$ and $DA = I_2$. So $D = A^{-1}$. Suppose $A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \end{bmatrix}$. How can we use D to solve for $\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$?

Block multiplication

Given A, an $m \times n$ matrix, and B, an $n \times p$ matrix, write $B = \begin{bmatrix} B_1 & B_2 & \cdots B_p \end{bmatrix}$, where the B_i are columns of B. Then

$$AB = A \begin{bmatrix} B_1 & B_2 & \cdots B_p \end{bmatrix} = \begin{bmatrix} AB_1 & AB_2 & \cdots & AB_p \end{bmatrix}.$$

Suppose A, B are both $n \times n$, and $AB = I_n$. Then

$$AB_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, AB_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, AB_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

Finding $B = A^{-1}$ amounts to solving (!) these systems of equations.

The inverse algorithm

Let A be an $n \times n$ matrix.

1. Form the augmented matrix $\left[\begin{array}{c|c} A & I_n \end{array}\right]$
2. Use row operations until you reach $\left[\begin{array}{c|c} R & B \end{array}\right]$, where R is the RREF of A.
3. If R has a row of zeros, then A is not invertible.
4. If $R = I_n$, then $B = A^{-1}$.

Examples

Compute the inverses (if possible) of:

- $$A = \begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix}$$
- $$B = \begin{bmatrix} 1 & 3 & -2 \\ -1 & -2 & 4 \\ 5 & 13 & -14 \end{bmatrix}$$
- $$C = \begin{bmatrix} 1 & 0 & -4 \\ 2 & 1 & -5 \\ 0 & 1 & 2 \end{bmatrix}$$

Invertible matrix theorem

Theorem:

Let A be an $n \times n$ matrix. The following statements are all equivalent to the statement "A is invertible":

1. The rank of A is n.
2. The RREF of A is equal to I_n.
3. The system $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
4. The *only* solution to the system $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
5. There is a matrix B such that $AB = I_n$.
6. There is a matrix C such that $CA = I_n$.

Properties of the inverse

Let A and B be invertible matrices.

1. The inverse of A is unique.
2. A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.
3. AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.
4. For any nonzero scalar k, kA is invertible, and $(kA)^{-1} = \frac{1}{k}A^{-1}$.
5. For any $n \times 1$ vector $\vec{b}$, the *unique* solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.

Examples

1. Show that if A, B, C are invertible, then $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
2. Show that if $A^3 = 4I_n$, then A is invertible.
3. Show that if $A^2 - 3A + 2I_n = 0$, then A is invertible.
4. Simplify the expression $B(AB)^{-1}(AB)^2B^{-1}$.