Eigenvalues and Eigenvectors
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Math 1410 Linear Algebra

Warm-up

- Recall: for any $n \times n$ matrix A, we get a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$.
- Each input vector $\vec{x}$ gets transformed to some output vector $\vec{y}$.
- Usually both the the direction and magnitude of $\vec{y} = A\vec{x}$ are different.

Now a warm-up exercise:

Let $A = \begin{bmatrix} 6 & 2 \\ -6 & -1 \end{bmatrix}$, and define $T(\vec{x}) = A\vec{x}$.

1. Compute $T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right)$
2. Compute $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$
3. Reflect on the results.

Multiplication operators

For any scalar λ, we can consider the function $f_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_\lambda(\vec{x}) = \lambda\vec{x}$.

This is just scalar multiplication.
Note f_λ is also a matrix transformation: $f_\lambda = f_A$, where

$$A = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} = \lambda I_n,$$

where I_n is the $n \times n$ identity matrix.

Invariant directions for matrix transformations

For the matrix $A = \lambda I_n$, $A\vec{x}$ is always parallel to $\vec{x}$.
For other matrices, there may be no vectors with this property: for example, if A is the rotation matrix in $\mathbb{R}^2$ (unless θ is a multiple of 2π).

In other cases, $A\vec{x}$ is parallel to $\vec{x}$ for some vectors but not others. For example, the shear

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Recommended reference: Understanding Linear Algebra by David Austin.

Eigenvalues and eigenvectors

Definition

For any $n \times n$ matrix A, we say that a scalar λ is an eigenvalue for A if there exists a non-zero vector $\vec{x}$, called an eigenvector, such that

$$A\vec{x} = \lambda\vec{x}$$

Example

Show that $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors of $A = \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$.

Eigenvectors and characteristic directions

Theorem

Suppose $\vec{x}$ is an eigenvector of an $n \times n$ matrix A, with eigenvalue λ. Then for any scalar $k \neq 0$, $k\vec{x}$ is also an eigenvector of A, corresponding to the same eigenvalue.

Finding eigenvalues and eigenvectors

Let A be an $n \times n$ matrix, and suppose λ is an eigenvalue of A. Then there exists a non-zero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$. That is, $\vec{x} \neq 0$ and

$$A\vec{x} - \lambda\vec{x} = (A - \lambda I_n)\vec{x} = 0.$$

Theorem

A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$

The characteristic polynomial

Definition

For any $n \times n$ matrix A, we define its characteristic polynomial by

$$c_A(x) = \det(xI_n - A).$$

Note: The eigenvalues of A are precisely the zeros of the characteristic polynomial of A.

Finding the characteristic polynomial

Example

Find the characteristic polynomial of $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 3 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

Examples

Find the eigenvalues of:

1. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
2. $B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$
3. $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Finding eigenvectors

- Suppose λ is an eigenvalue of A.
- Then $A\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$.
- That means $\vec{v}$ is a solution to $(A - \lambda I)\vec{x} = \vec{0}$.

So we find eigenvectors by solving homogeneous systems!

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$.

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ 5 & 1 & 3 \end{bmatrix}$.

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & -1 \\ 10 & 8 & -3 \end{bmatrix}$

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.