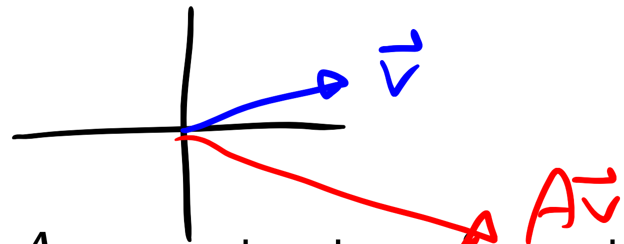
Eigenvalues and Eigenvectors
March 26, 2020

Math 1410 Linear Algebra

Warm-up



- Recall: for any $n \times n$ matrix A, we get a transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T(\vec{x}) = A\vec{x}$.
- Each input vector $\vec{x}$ gets transformed to some output vector $\vec{y}$.
- Usually both the the direction and magnitude of $\vec{y} = A\vec{x}$ are different.

$T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 8 \\ -7 \end{bmatrix} \to$ not parallel to $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Now a warm-up exercise:

Let $A = \begin{bmatrix} 6 & 2 \\ -6 & -1 \end{bmatrix}$, and define $T(\vec{x}) = A\vec{x}$.

1. Compute $T\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 6 & 2 \\ -6 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 6-4 \\ -6+2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$
2. Compute $T\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 6 & 2 \\ -6 & -1 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -12+6 \\ 12-3 \end{bmatrix} = \begin{bmatrix} -6 \\ 9 \end{bmatrix} = 3\begin{bmatrix} -2 \\ 3 \end{bmatrix}$
3. Reflect on the results.

Multiplication operators

For any scalar λ, we can consider the function $f_\lambda : \mathbb{R}^n \to \mathbb{R}^n$ given by $f_\lambda(\vec{x}) = \lambda\vec{x}$.

This is just scalar multiplication.
Note f_λ is also a matrix transformation: $f_\lambda = f_A$, where

$$A = \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{bmatrix} = \lambda I_n,$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

$$D\hat{\imath} = 2\hat{\imath}$$

$$D\hat{\jmath} = 3\hat{\jmath}$$

$$D\hat{k} = -4\hat{k}$$

where I_n is the $n \times n$ identity matrix.

$$A\vec{x} = (\lambda I_n)\vec{x} = \lambda(I_n\vec{x}) = \lambda\vec{x}$$

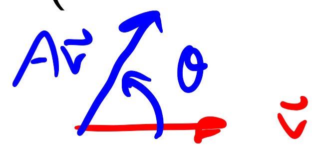
for <u>any</u> $\vec{x}$.

Invariant directions for matrix transformations

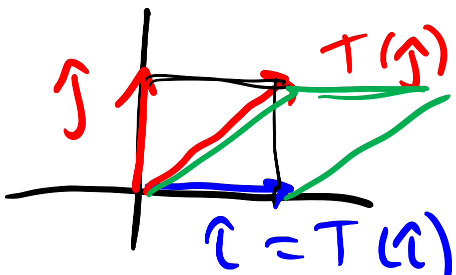
For the matrix $A = \lambda I_n$, $A\vec{x}$ is always parallel to $\vec{x}$.

For other matrices, there may be no vectors with this property: for example, if A is the rotation matrix in $\mathbb{R}^2$ (unless θ is a multiple of 2π).

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



In other cases, $A\vec{x}$ is parallel to $\vec{x}$ for some vectors but not others. For example, the shear



$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Recommended reference: Understanding Linear Algebra by David Austin.

Eigenvalues and eigenvectors

lambda

Definition

For any $n \times n$ matrix A, we say that a scalar λ is an eigenvalue for A if there exists a non-zero vector $\vec{x}$, called an eigenvector, such that

$$\vec{y} = A\vec{x} = \lambda\vec{x}$$

Example

Show that $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ are eigenvectors of $A = \begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$.

$$\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 8-4 \\ -4+5 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \lambda = 1 \quad \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

$A\vec{x}$

$$\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2+4 \\ -1-5 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix} \quad \lambda = 6.$$

Eigenvectors and characteristic directions

Theorem

Suppose $\vec{x}$ is an eigenvector of an $n \times n$ matrix A, with eigenvalue λ. Then for any scalar $k \neq 0$, $k\vec{x}$ is also an eigenvector of A, corresponding to the same eigenvalue.

Proof: Let $\vec{x}$ be an eigenvector with eigenvalue λ. Then $\boxed{A\vec{x} = \lambda\vec{x}}$.

$$\therefore A(k\vec{x}) = k(A\vec{x}) = k(\lambda\vec{x}) = \lambda(k\vec{x})$$

$(k \neq 0)$

Finding eigenvalues and eigenvectors

Let A be an $n \times n$ matrix, and suppose λ is an eigenvalue of A. Then there exists a non-zero vector $\vec{x}$ such that $\boxed{A\vec{x} = \lambda\vec{x}.}$ That is, $\vec{x} \neq 0$ and

$$A\vec{x} - \lambda\vec{x} = (A - \lambda I_n)\vec{x} = \vec{0}.$$

$$\underbrace{\boxed{(A - \lambda I)\vec{x} = \vec{0}}}_{\text{homogeneous system}}, \quad \vec{x} \neq \vec{0}$$

Non-trivial solutions if and only if $A - \lambda I$ is not invertible

$\therefore \det(A - \lambda I) = 0$

Theorem

A scalar $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $\det(A - \lambda I_n) = 0$

The characteristic polynomial

Definition

For any $n \times n$ matrix A, we define its characteristic polynomial by

$$c_A(x) = \det(xI_n - A). = (-1)^n \det(A - xI)$$

(not $A - xI$)

Note: The eigenvalues of A are precisely the zeros of the characteristic polynomial of A.

$$c_A(x) = x^n - a_{n-1}x^{n-1} + \dots + a_1 x + a_0$$

– find zeros by factoring!

$$c_A(x) = (x-\lambda_1)^{m_1}(x-\lambda_2)^{m_2} \cdots (x-\lambda_k)^{m_k}$$

Finding the characteristic polynomial

Example

Find the characteristic polynomial of $A = \begin{bmatrix} 2 & 3 & -1 \\ 0 & 3 & 2 \\ 1 & 2 & 0 \end{bmatrix}$.

$$c_A(x) = \det(xI - A)$$

$$= \begin{vmatrix} x-2 & -3 & 1 \\ 0 & x-3 & -2 \\ -1 & -2 & x \end{vmatrix} = (x-2)\begin{vmatrix} x-3 & -2 \\ -2 & x \end{vmatrix} + 0 - 1\begin{vmatrix} -3 & 1 \\ x-3 & -2 \end{vmatrix}$$

$$= (x-2)(\underline{x^2 - 3x} - 4) - 1\left(\overset{9-x}{6 - (x-3)}\right)$$

$$= x^3 - 3x^2 - 4x - 2x^2 + 6x + 8 + x - 9$$

$$= x^3 - 5x^2 + 3x - 1.$$

Examples

Find the eigenvalues of:

1. $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
2. $B = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$
3. $C = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ – rotation by $90°$

$$\begin{vmatrix} x-2 & -1 \\ -1 & x-2 \end{vmatrix} = (x-2)^2 - 1 = x^2-4x+4-1 = x^2-4x+3 = (x-3)(x-1)$$

$$= ((x-2)-1)((x-2)+1)$$

$$= (x-3)(x-1)$$

$\lambda = 1$, $\lambda = 3$.

$$C_B(x) = \det(xI - B) = \begin{vmatrix} x-3 & -1 \\ 0 & x-3 \end{vmatrix} = (x-3)^2$$

$\lambda = 3$ (twice)

$$C_C(x) = \begin{vmatrix} x & +1 \\ -1 & x \end{vmatrix} = x^2+1 = (x-i)(x+i)$$

no real eigenvalues

$\lambda = \pm i = \pm\sqrt{-1}$

Finding eigenvectors

$$A - 3I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

- Suppose λ is an eigenvalue of A.
- Then $A\vec{v} = \lambda\vec{v}$ for some nonzero vector $\vec{v}$.
- That means $\vec{v}$ is a solution to $(A - \lambda I)\vec{x} = \vec{0}$.

So we find eigenvectors by solving homogeneous systems!

Eg: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\lambda = 1$, $\lambda = 3$.

For $\lambda = 1$: solve $(A - 1I)\vec{x} = \vec{0}$

$$A - I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x + y = 0 \implies y = -x$$

$$\vec{v} = \begin{bmatrix} x \\ -x \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

put $x=1$ $\therefore$ $\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Example

$A\vec{v} = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}$ $A\vec{w} = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & -5 \\ -5 & 1 \end{bmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & -5 \\ -5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 25$$

$$= \lambda^2 - 2\lambda + 1 - 25$$

$$= \lambda^2 - 2\lambda - 24$$

$$= (\lambda - 6)(\lambda + 4)$$

$\therefore$ Eigenvalues are $\lambda = 6$, $\lambda = -4$.

For $\lambda = 6$: $A - 6I = \begin{bmatrix} -5 & -5 \\ -5 & -5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $x + y = 0$

$1-6$

$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

For $\lambda = -4$: $A + 4I = \begin{bmatrix} 5 & -5 \\ -5 & 5 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x - y = 0$

$\vec{w} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ 5 & 1 & 3 \end{bmatrix}$.

– Exercise!

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & -1 \\ 10 & 8 & -3 \end{bmatrix}$

Example

Note: $A^T = A$

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$.

$$C_A(x) = \det(xI - A) = \begin{vmatrix} x & -1 & -1 \\ -1 & x & -1 \\ -1 & -1 & x \end{vmatrix} = x \begin{vmatrix} x & -1 \\ -1 & x \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ -1 & x \end{vmatrix} - 1 \begin{vmatrix} -1 & -1 \\ x & -1 \end{vmatrix}$$

$+$, $-$, $+$; $-(x+1)$

$$= x(x^2-1) + (-x-1) - (x+1)$$

$$= x(x-1)(x+1) - 2(x+1)$$

$$= (x+1)(x(x-1)-2) = (x+1)(x^2-x-2) = (x+1)(x-2)(x+1)$$

$$= (x+1)^2(x-2)$$

$\lambda = -1$, $\lambda = 2$.

$\lambda = -1$: $A - (-1)I = A + I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad x + y + z = 0$

$x = -y - z$

(over to next page)

$\lambda = -1$: $\quad x + y + z = 0$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$y = 1,\ z = 0$ gives $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$y = 0,\ z = 1$ gives $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\lambda = 2$: $\quad A - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\qquad x = z$, $y = z$ $\qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 - R_1 \to R_2 \\ R_3 + 2R_1 \to R_3 \end{matrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$