Math 1410, Spring 2020

Matrix Transformations

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Overview

Warm-Up

1. Show that if $A^3 = 4I_n$, then A is invertible.
2. Show that if $A^2 - 3A + 2I_n = 0$, then A is invertible.
3. Simplify the expression $B(AB)^{-1}(AB)^2B^{-1}$.
4. Find the inverse of $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}$.

Invertible matrix theorem

Theorem:

Let A be an $n \times n$ matrix. The following statements are all equivalent to the statement "A is invertible":

1. The rank of A is n.
2. The RREF of A is equal to I_n.
3. The system $A\vec{x} = \vec{b}$ is consistent for every $n \times 1$ vector $\vec{b}$.
4. The *only* solution to the system $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
5. There is a matrix B such that $AB = I_n$.
6. There is a matrix C such that $CA = I_n$.

Properties of the inverse

Let A and B be invertible matrices.

1. The inverse of A is unique.
2. A^{-1} is also invertible, and $(A^{-1})^{-1} = A$.
3. AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.
4. For any nonzero scalar k, kA is invertible, and $(kA)^{-1} = \frac{1}{k}A^{-1}$.
5. For any $n \times 1$ vector $\vec{b}$, the *unique* solution to $A\vec{x} = \vec{b}$ is $\vec{x} = A^{-1}\vec{b}$.

Matrix-vector products, again

Recall: if A is an $m \times n$ matrix, and $\vec{x}$ is an $n \times 1$ vector, then $\vec{y} = A\vec{x}$ is an $m \times 1$ vector. Let $\mathbb{R}^n$ (sometimes, $\mathbb{R}^{n,1}$) denote the set of all $n \times 1$ column vectors. Then any $m \times n$ matrix A can be used to define a *function*

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

defined by $T(\vec{x}) = A\vec{x}$. What properties does this function have?

Examples

Let $A = \begin{bmatrix} 3 & -2 \\ 5 & 4 \end{bmatrix}$, and let $T(\vec{x}) = A\vec{x}$. Compute the value of T on:

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Repeat the above for $A = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix}$.

Properties

Theorem:

Let A be an $m \times n$ matrix, and define $T(\vec{x}) = A\vec{x}$. Then:

1. T is a function from $\mathbb{R}^n$ to $\mathbb{R}^m$
2. $T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y})$ for all $\vec{x}, \vec{y}$
3. $T(c\vec{x}) = cT(\vec{x})$ for all vectors $\vec{x}$ and scalars c

Note: any function between vector spaces (e.g. $\mathbb{R}^n, \mathbb{R}^m$) with these properties is called a linear transformation. It turns out any linear transformation can be expressed as a matrix transformation.

Examples

1. Show that for any linear transformation, $T(\vec{0}) = \vec{0}$.
2. Given $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\-2\end{bmatrix}$ and $T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-4\\5\end{bmatrix}$, find $T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)$
3. Given $T(\vec{a}) = \begin{bmatrix}1\\0\\-3\end{bmatrix}$, $T(\vec{b}) = \begin{bmatrix}0\\-2\\5\end{bmatrix}$ and $T(\vec{c}) = \begin{bmatrix}1\\1\\1\end{bmatrix}$, find $T(2\vec{a} - 3\vec{b} + 5\vec{c})$.

Examples

1. If $T(\vec{x}) = A\vec{x}$ for $A = \begin{bmatrix} 2 & -3 \\ -1 & -2 \\ 5 & 4 \end{bmatrix}$, what are the domain and codomain of T?
2. For T as above, compute $T(\hat{\imath})$ and $T(\hat{\jmath})$
3. What if $A = \begin{bmatrix} 2 & -1 & 5 \\ 4 & -2 & 3 \end{bmatrix}$?
4. If $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 3x - 2y \\ -x + 4y + 5z \\ 7x - 2y - 6z \end{bmatrix}$, for what matrix is $T(\vec{x}) = A\vec{x}$?

Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

When A is 2×2 we can visualize everything in terms of geometric vectors in the plane. We can use matrices to describe *transformations*, like stretches, rotations, and reflections. (But not translations.)

Example

Describe the effect of the transformation with matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$ in terms of what it does to the *unit square* $(0 \leq x, y \leq 1)$

Transformation matrices

- *Stretches*: $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$, $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} = kI_2$. (This is just scalar multiplication.)
- *Reflections*: $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- *Rotations*: $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$
- *Shears*: $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Examples

Determine the matrix transformation that:

1. Stretches horizontally by a factor of 2, rotates by 90°, and then reflects across the x axis.
2. Reflects across the line $y = x$, stretches vertically be a factor of 3, then reflects across the y axis.