Properties of Eigenvalues and Eigenvectors
March 31, 2020

Math 1410 Linear Algebra

Warm-up

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ 5 & 1 & 3 \end{bmatrix}$.

Eigenvalues and eigenvectors

Definition

For any $n \times n$ matrix A, we say that a scalar λ is an eigenvalue for A if there exists a non-zero vector $\vec{x}$, called an eigenvector, such that

$$A\vec{x} = \lambda\vec{x}$$

Definition

For any $n \times n$ matrix A, we define its characteristic polynomial by

$$c_A(x) = \det(xI_n - A).$$

The eigenvalues of A are the roots of $c_A(x)$.
The eigenvector(s) corresponding to an eigenvalue λ are the basic solutions to $(A - \lambda I)\vec{x} = \vec{0}$.

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & -1 \\ 10 & 8 & -3 \end{bmatrix}$

Case of triangular matrices

Let A be an upper-triangular $n \times n$ matrix.
What can we say about the trace, determinant, and eigenvalues of A? What about eigenvectors?
Examples:

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & -5 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Similar matrices

Definition

We say that two matrices A and B are similar if there exists an invertible matrix P such that $B = P^{-1}AP$.

Theorem

If A and B are similar matrices, then they have the same trace, determinant, and eigenvalues.

Triangularization

Theorem

Every $n \times n$ matrix A is similar to a triangular matrix.

Note #1: you might think that this is our ticket to easily finding eigenvalues. But no – the process for finding a triangular matrix similar to A typically involves finding eigenvalues and eigenvectors. (And in some cases *generalized* eigenvectors!)

Note #2: this theorem isn't quite true if we work only over the real numbers.

Diagonalization

Definition

We say that an $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix; that is, if there exists an invertible matrix P such that $D = P^{-1}AP$ is diagonal.

Note: since similar matrices have the same eigenvalues, we must have

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Theorem

An $n \times n$ matrix A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Example

Determine whether or not the matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{bmatrix}$ can be diagonalized.

Case of distinct eigenvalues

Theorem
If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of a matrix A, then the corresponding eigenvectors $\vec{x}_1, \ldots, \vec{x}_m$ are linearly independent.

Fact: Any set of n linearly independent vectors forms a basis of $\mathbb{R}^n$.

Repeated eigenvalues

In general a matrix A will have characteristic polynomial

$$c_A(x) = (x - \lambda_1)^{k_1}(x - \lambda_2)^{k_2} \cdots (x - \lambda_m)^{k_m},$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues and $k_1, \ldots, k_m$ are the multiplicities of the eigenvalues.

Definition

Given an eigenvalue λ of a matrix A, we define the eigenspace $E(\lambda, A)$ of A with respect to λ by

$$E(\lambda, A) = \{\vec{x} \,|\, (A - \lambda I_n)\vec{x} = 0\} = \text{null}(A - \lambda I).$$

Note: we always have $1 \leq \dim E(\lambda_j, A) \leq k_j$ for each j.
A matrix A is diagonalizable if and only if $\dim E(\lambda_j, A) = k_j$ for each $j = 1, 2, \ldots, k$.

Example

Determine whether or not the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & 3 \\ -1 & 2 & 2 \end{bmatrix}$ is diagonalizable.

Example

Determine whether or not the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ -2 & -2 & 2 \\ -5 & -10 & 7 \end{bmatrix}$ is diagonalizable.

Powers of matrices

Suppose we wanted to find A^7, where A was the matrix from the last slide. Finding this by hand would take a very long time. (For large matrices and high powers, even a computer will take a long time.)

However, we know that $A = PDP^{-1}$, where $P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$.

Polynomials of matrices

Suppose $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial and we want to compute $p(A)$, where A is diagonalizable.

Symmetric matrices

Recall: an $n \times n$ matrix A is symmetric if $A^T = A$.

Theorem

Suppose A is a symmetric matrix. If $\vec{x}_1$ and $\vec{x}_2$ are eigenvalues of A corresponding to eigenvalues $\lambda_1 \neq \lambda_2$, then $\vec{x}_1 \cdot \vec{x}_2 = 0$.

Theorem

If A is an $n \times n$ symmetric matrix, then there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Example

Given $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, find an orthogonal matrix P such that P^TAP is diagonal.

Example

Sketch the curve defined by the equation $3x^2 + 2xy + 3y^2 = 1$.