Properties of Eigenvalues and Eigenvectors
March 31, 2020

Math 1410 Linear Algebra

Warm-up

$$xI = \begin{bmatrix} x & 0 & 0 \\ 0 & x & 0 \\ 0 & 0 & x \end{bmatrix}$$

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ 5 & 1 & 3 \end{bmatrix}$.

$$c_A(x) = \det(xI - A)$$
$$= \begin{vmatrix} x-3 & 0 & 0 \\ -1 & x-1 & 1 \\ -5 & -1 & x-3 \end{vmatrix} = (x-3)\begin{vmatrix} x-1 & 1 \\ -1 & x-3 \end{vmatrix}$$
$$= (x-3)\big((x-1)(x-3)+1\big)$$
$$= (x-3)(x^2-4x+3+1)$$
$$= (x-3)(x^2-4x+4) = (x-3)(x-2)^2$$

Find using computer

$\lambda = 3$, $\underbrace{\lambda = 2}$ (twice)

"multiplicity 2"

$\underline{\lambda = 3}$: $(A-3I) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & -1 \\ 5 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 11 & 5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/11 \\ 0 & 1 & 5/11 \\ 0 & 0 & 0 \end{bmatrix}$

$3I - A = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 1 \\ -5 & -1 & 0 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $(A-3I)\vec{x} = \vec{0}$ $\Rightarrow$ $x - \frac{1}{11}z = 0$

$y + \frac{5}{11}z = 0$

$\therefore$ For $\lambda = 3$, $\quad \vec{x}_1 = \begin{bmatrix} \frac{1}{11}z \\ -\frac{5}{11}z \\ z \end{bmatrix}$ For $z = 11$ $\quad \vec{x} = \begin{bmatrix} 1 \\ -5 \\ 11 \end{bmatrix}$

Check: $\quad A\vec{x} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 1 & -1 \\ 5 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -5 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ -15 \\ 33 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -5 \\ 11 \end{bmatrix}.$

$\lambda = 2$: $\quad A - 2I = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 5 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$\vec{x}_2 = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad (A - 2I)\vec{x} = \vec{0} \implies x = 0$

$y + z = 0$

$\vec{x}_2 = \begin{bmatrix} 0 \\ -z \\ z \end{bmatrix} \quad z = 1$ gives $\vec{x}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

Eigenvalues and eigenvectors

Definition

For any $n \times n$ matrix A, we say that a scalar λ is an eigenvalue for A if there exists a non-zero vector $\vec{x}$, called an eigenvector, such that

$$A\vec{x} = \lambda\vec{x}$$

i.e. $\underline{(A - \lambda I)}\vec{x} = \vec{0}$

Definition

For any $n \times n$ matrix A, we define its characteristic polynomial by

$$c_A(x) = \underbrace{\det(xI_n - A)}_{\text{polynomial}}. = 0$$

The eigenvalues of A are the roots of $c_A(x)$.
The eigenvector(s) corresponding to an eigenvalue λ are the basic solutions to $(A - \lambda I)\vec{x} = \vec{0}$.

Example

Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 4 & 2 & -1 \\ 2 & 3 & -1 \\ 10 & 8 & -3 \end{bmatrix}$

$$C_A(x) = \det(xI - A) = \begin{vmatrix} x-4 & -2 & 1 \\ -2 & x-3 & 1 \\ -10 & -8 & x+3 \end{vmatrix}$$

$$= (x-4)\begin{vmatrix} x-3 & 1 \\ -8 & x+3 \end{vmatrix} \overset{(-2)(-1)^{1+2}}{+2} \begin{vmatrix} -2 & 1 \\ -10 & x+3 \end{vmatrix} + 1 \begin{vmatrix} -2 & x-3 \\ -10 & -8 \end{vmatrix}$$

$$= (x-4)(x^2 - 9 + 8) + 2(\overset{-2x+4}{-2x-6+10}) + (16 + 10x - 30)$$

$$= (x-4)(x^2-1) + \underbrace{-4(x-2) + 10x - 14}_{x^2-3x+2}$$

$$= (x-4)(x-1)(x+1) + 6(x-1) = (x-1)(x^2-3x-4+6)$$

$$= (x-1)(x-1)(x-2)$$

$\lambda = 1, \lambda = 2.$

$\lambda = 1$ (multiplicity 2)

$$A - 1I = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 3 & -1 \\ 10 & 8 & -3 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - I = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & -1 \\ 10 & 8 & -4 \end{bmatrix} \quad \begin{matrix} R_1 - R_2 \to R_1 \\ R_3 - 5R_2 \to R_3 \end{matrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & -2 & 1 \end{pmatrix}$$

$$\xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$(A - I)\vec{x} = 0$

and $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow x = 0$

$2y - z = 0$

$$\vec{x} = \begin{bmatrix} 0 \\ y \\ 2y \end{bmatrix} \qquad y = 1 \Rightarrow \vec{x} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{1}{2}z \\ z \end{bmatrix} \quad (z = 2)$$

$\lambda = 2$: exercise.

$$A\vec{x} = \begin{pmatrix} 4 & 2 & -1 \\ 2 & 3 & -1 \\ 10 & 8 & -3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \checkmark$$

Case of triangular matrices

Let A be an upper-triangular $n \times n$ matrix.
What can we say about the trace, determinant, and eigenvalues of A? What about eigenvectors?
Examples:

only 1 eigenvector

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 4 & -2 \\ 0 & 0 & -5 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \quad \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \underbrace{\vec{e}_1, \vec{e}_2}_{\lambda=2}, \underbrace{\vec{e}_3}_{\lambda=3}$$

$\det(A_1) = -60$ $\quad \det(A_2) = 12$ $\quad \det(A_3) = 8$ $\quad \det(A_4) = 12$

$$C_{A_1}(x) = \begin{vmatrix} x-3 & -2 & -1 \\ 0 & x-4 & 2 \\ 0 & 0 & x+5 \end{vmatrix}$$

$$= (x-3)(x-4)(x+5)$$

$$\lambda = 3, 4, -5$$

$$A_1 - 3I = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & -8 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Eigenvector $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Similar matrices

$A \sim B \Longleftrightarrow B \sim A$

Definition

We say that two matrices A and B are similar if there exists an invertible matrix P such that $B = P^{-1}AP$.

$PB = AP$ $\quad PBP^{-1} = A$

$Q^{-1}BQ = A \quad (Q = P^{-1})$

Theorem

If A and B are similar matrices, then they have the same trace, determinant, and eigenvalues.

$$|B| = |P^{-1}AP| = |P^{-1}||A||P| = \frac{1}{\cancel{|P|}}|A|\cancel{|P|}$$

$$P^{-1}(A - \lambda I)P = P^{-1}AP - P^{-1}(\lambda I)P$$
$$= B - \lambda I$$

$$B - \lambda I \sim A - \lambda I$$

Triangularization

can find P – invertible
B – triangular
$B = P^{-1}AP$

Theorem

Every $n \times n$ matrix A is similar to a triangular matrix.

Note #1: you might think that this is our ticket to easily finding eigenvalues. But no – the process for finding a triangular matrix similar to A typically involves finding eigenvalues and eigenvectors. (And in some cases *generalized* eigenvectors!)

Note #2: this theorem isn't quite true if we work only over the real numbers.

Diagonalization

Eg: want A^{10}, and $A = PDP^{-1}$

$$A^{10} = \underbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\cancel{\cdots}\cancel{}(\cancel{P}DP^{-1})}_{10 \text{ times}} = PD^{10}P^{-1}$$

Definition

We say that an $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix; that is, if there exists an invertible matrix P such that $D = P^{-1}AP$ is diagonal.

Note: since similar matrices have the same eigenvalues, we must have

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

$$D^{10} = \begin{pmatrix} \lambda_1^{10} & & \\ & \lambda_2^{10} & \\ & & \ddots \end{pmatrix}$$

Theorem

An $n \times n$ matrix A is diagonalizable if and only if there exists a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

Example

Determine whether or not the matrix $A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & -1 \\ 1 & -1 & 4 \end{bmatrix}$ can be diagonalized.

$$C_A(x) = \begin{vmatrix} x-2 & 0 & -2 \\ 0 & x-2 & 1 \\ -1 & 1 & x-4 \end{vmatrix} = (x-2)\begin{vmatrix} x-2 & 1 \\ 1 & x-4 \end{vmatrix} - 1 \begin{vmatrix} 0 & -2 \\ x-2 & 1 \end{vmatrix}$$

$$= (x-2)(x^2-6x+8-1) + 1(-2)(x-2)$$

$$= (x-2)(x^2-6x+7-2)$$

$$= (x-2)(x^2-6x+5)$$

$$= \underbrace{(x-2)}_{\vec{x}_1}\underbrace{(x-1)}_{\vec{x}_2}\underbrace{(x-5)}_{\vec{x}_3}$$

$$P = \begin{bmatrix} | & | & | \\ \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \\ | & | & | \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} = D$$

~~Yes~~ : $\lambda = 1, 2, 5$
each gets an eigenvector.

Case of distinct eigenvalues

Theorem

If $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of a matrix A, then the corresponding eigenvectors $\vec{x}_1, \ldots, \vec{x}_m$ are linearly independent.

Fact: Any set of n linearly independent vectors forms a basis of $\mathbb{R}^n$.

Repeated eigenvalues

In general a matrix A will have characteristic polynomial

$$c_A(x) = (x - \lambda_1)^{k_1}(x - \lambda_2)^{k_2} \cdots (x - \lambda_m)^{k_m},$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues and $k_1, \ldots, k_m$ are the multiplicities of the eigenvalues.

Definition

Given an eigenvalue λ of a matrix A, we define the eigenspace $E(\lambda, A)$ of A with respect to λ by

$$E(\lambda, A) = \{\vec{x} \,|\, (A - \lambda I_n)\vec{x} = 0\} = \text{null}(A - \lambda I).$$

Note: we always have $1 \leq \dim E(\lambda_j, A) \leq k_j$ for each j.
A matrix A is diagonalizable if and only if $\dim E(\lambda_j, A) = k_j$ for each $j = 1, 2, \ldots, k$.

Example

Determine whether or not the matrix $A = \begin{bmatrix} 1 & 2 & 0 \\ -3 & 2 & 3 \\ -1 & 2 & 2 \end{bmatrix}$ is diagonalizable.

– try it: post on the forum!

Example

Determine whether or not the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ -2 & -2 & 2 \\ -5 & -10 & 7 \end{bmatrix}$ is diagonalizable.

Powers of matrices

Suppose we wanted to find A^7, where A was the matrix from the last slide. Finding this by hand would take a very long time. (For large matrices and high powers, even a computer will take a long time.)

However, we know that $A = PDP^{-1}$, where $P = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 5 \end{bmatrix}$.

Polynomials of matrices

Suppose $p(x) = a_n x^n + \cdots + a_1 x + a_0$ is a polynomial and we want to compute $p(A)$, where A is diagonalizable.

Symmetric matrices

Recall: an $n \times n$ matrix A is symmetric if $A^T = A$.

Theorem

Suppose A is a symmetric matrix. If $\vec{x}_1$ and $\vec{x}_2$ are eigenvalues of A corresponding to eigenvalues $\lambda_1 \neq \lambda_2$, then $\vec{x}_1 \cdot \vec{x}_2 = 0$.

Theorem

If A is an $n \times n$ symmetric matrix, then there exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

$\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ $\qquad A\vec{x}_i = \lambda_i \vec{x}_i$

$$\vec{x}_i \cdot \vec{x}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \quad (\|\vec{x}_i\| = 1)$$

using these to P, $\quad P^{-1} = P^T$ (P is orthogonal)

Example

Given $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, find an orthogonal matrix P such that $P^T A P$ is diagonal.

Example

Sketch the curve defined by the equation $3x^2 + 2xy + 3y^2 = 1$.