Math 1410, Spring 2020

Introduction to Vectors

January 24, 2020

Overview

1. Geometric vectors

2. Algebra of vectors

Cartesian coordinates

We will work mainly in two and three dimensions. We write:

- $\mathbb{R}^2 = \{(x, y) \,|\, x, y \in \mathbb{R}\}$ — set of all points in the plane
- $\mathbb{R}^3 = \{(x, y, z) \,|\, x, y, z \in \mathbb{R}\}$ — set of all points in "space"
- Can define $\mathbb{R}^4, \mathbb{R}^5$, etc. similarly, but can't draw them.

Distance comes from the **Pythagorean theorem**. In $\mathbb{R}^2$, the distance from (x_1, y_1) to (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

In $\mathbb{R}^3$, we just have to add the z-coordinate:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Vectors

A **vector** is a "directed line segment". It has *magnitude* and *direction*. The *position* of a vector usually doesn't matter. Given points P and Q we write $\vec{v} = \overrightarrow{PQ}$ for the vector *from* P *to* Q. (Tail at P, tip at Q.) The **magnitude** of $\vec{v}$ is just the distance from tip to tail. We write $\|\vec{v}\|$ for the magnitude of $\vec{v}$.

Vector notation

We can't do much by just drawing arrows. Let $P = (x_1, y_1), Q = (x_2, y_2)$ be two points in the plane. We write $\vec{v} = \overrightarrow{PQ}$ in *component form* as $\vec{v} = \langle x_2 - x_1, y_2 - y_1 \rangle$. Equivalent notation is the *column vector* $\vec{v} = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$. Magnitude becomes $\|\vec{v}\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

Examples

Let $P = (1,2)$ and $Q = (3,-4)$. Find:

1. $\overrightarrow{PQ}$
2. $\overrightarrow{QP}$
3. $\left\|\overrightarrow{PQ}\right\|$

Exercise: Repeat the above for $P = (1,-2,4)$ and $Q = (3,0,-2)$.
Exercise: If $\overrightarrow{PQ} = \langle 3,-1,5\rangle$ and $Q = (2,2,-3)$, what is P?

Contexts

Vectors appear frequently in Physics:

- Displacement
- Velocity
- Force
- etc.

Can think in terms of simple directions: $\vec{v} = \langle 3, 4 \rangle$ tells us to go 3 units right, 4 units up. Also find vectors as data arrays (but no longer geometric). Meaning of "vector" expands in second linear algebra course.

Adding vectors

To add vectors, we simply add components.

- In $\mathbb{R}^2$, with $\vec{v} = \langle v_1, v_2 \rangle, \vec{w} = \langle w_1, w_2 \rangle$,

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2 \rangle .$$

- In $\mathbb{R}^3$, with $\vec{v} = \langle v_1, v_2, v_3 \rangle, \vec{w} = \langle w_1, w_2, w_3 \rangle$,

$$\vec{v} + \vec{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle .$$

Exercise:

1. Ask me for examples.
2. Find $\langle 2, 4 \rangle + \langle -5, 1 \rangle$.
3. Find $\langle 4, -2, 1 \rangle - \langle 3, -7, 6 \rangle$. (How do you think we should define subtraction?)
4. If $\vec{x} + \langle 3, 2 \rangle = \langle 5, 9 \rangle$, what is $\vec{x}$?

Addition, geometrically

As "arrows", vector addition follows "tip-to-tail rule":

1. Draw $\vec{v}$.
2. Draw $\vec{w}$ with its tail at the tip of $\vec{v}$.
3. Draw an arrow from the tail of $\vec{v}$ to the tip of $\vec{w}$. This is $\vec{v} + \vec{w}$.

Exercise: try this, for $\vec{v} = \langle 2, 1 \rangle$ and $\vec{w} = \langle 3, 2 \rangle$. Exercise: for $P = (1, 0), Q = (2, 2), R = (4, 5)$, compute $\overrightarrow{PQ}$, $\overrightarrow{QR}$, $\overrightarrow{PR}$, and $\overrightarrow{PQ} + \overrightarrow{QR}$.
Challenge: show why this works in general, for $P = (x_1, y_1), Q = (x_2, y_2), R = (x_3, y_3)$.

Scalar multiplication

- Exercise: if $\vec{v} = \langle 3, 2, 4 \rangle$, what is $\vec{v} + \vec{v}$? How are the components related to those of $\vec{v}$?
- We don't have a good way to define multiplication of two vectors. But we *can* multiply a vector by a number. This is called **scalar multiplication**. Given a real number c and a vector $\vec{v} = \langle a, b \rangle$,

$$c\vec{v} = c \langle a, b \rangle = \langle ca, cb \rangle .$$

The story in $\mathbb{R}^3$ is similar:

$$c \langle x, y, z \rangle = \langle cx, cy, cz \rangle .$$

Scalar multiplication, geometrically

- $2\vec{v} = \vec{v} + \vec{v}$, so $2\vec{v}$ is in the same direction as $\vec{v}$, but twice as long.
- What does $-2\vec{v}$ give us?
- In general, we have:

Theorem:

For any vector $\vec{v}$ and scalar (number) c,

$$\|c\vec{v}\| = |c|\,\|\vec{v}\|.$$

Parallel vectors, unit vectors

- **Definition: Parallel vectors.**

  We say that two vectors $\vec{v}$ and $\vec{w}$ are **parallel** if $\vec{w} = c\vec{v}$ for some scalar c.

- **Definition: Unit vector.**

  A vector $\vec{u}$ is a **unit vector** if $\|\vec{u}\| = 1$.

- Unit vectors are useful when we care about direction, but not magnitude.
- Given $\vec{v} = \langle 2, 3 \rangle$, what is a unit vector in the direction of $\vec{v}$?

Standard unit vectors

In $\mathbb{R}^2$:

$$\hat{\imath} = \langle 1, 0 \rangle \,, \quad \hat{\jmath} = \langle 0, 1 \rangle \,.$$

In $\mathbb{R}^3$:

$$\hat{\imath} = \langle 1, 0, 0 \rangle \,, \quad \hat{\jmath} = \langle 0, 1, 0 \rangle \,, \quad \hat{k} = \langle 0, 0, 1 \rangle \,.$$