Math 2565, Spring 2020

Functions of several variables (a rather rapid introduction)

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Overview

Surfaces -- recap

Two basic representations:

- Graphs: $z = f(x, y)$
- Level sets: $g(x, y, z) = c$
- Plotting utilities recommended. CalcPlot3D is a good online resource for this.

Functions of two variables

A function f of two variables takes a point $(x, y) \in \mathbb{R}^2$ as an input, and gives a point $z = f(x, y)$ as an output.

If $D \subseteq \mathbb{R}^2$ is the domain of f we might write $f : D \to \mathbb{R}$ to emphasise the types of input and output.

Examples:

Examples

For the following functions, compute the values at $(0,0),(1,2)$, and $(3,-2)$. Determine the domain and range.

1. $f(x,y) = x^2 - y^2$
2. $f(x,y) = e^{-(x^2+y^2)}$
3. $f(x,y) = \ln(xy)$

Graphing functions of two variables

How do we even begin to visualize a function of two variables?
(Or three, or four, or...)
The key is to consider traces of the graph $z = f(x, y)$.
Traces in planes $z = k$ are especially useful.
These are called *level curves*.
Examples:

$$z = x - y^2 \quad z = 4x^2 + y^2 \quad z = \sin(x)\cos(y)$$

Level curves vs. contour plots

You'll see mentions of both level curves and contour plots. These are related, but different.

A level curve lies *on the surface*, in a plane $z = k$.

A contour curve is the curve we get in the xy plane if we set $z = 0$.

A contour plot is a collection of contour curves $f(x, y) = k$ for different values of k.

Limits

Limits in more than one variable are tricky.

- In one variable, we need to ensure left and right hand limits agree.
- In two or more, we need to approach the same value *along any possible path*.

Example

Show that the following limits don't exist:

1. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2x+3y^2}{x+y}$
2. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$

Derivatives

The difficulty with limits makes defining "the" derivative a challenge:

We can't really generalize $f'(c) = \lim_{h\to 0} \dfrac{f(c+h) - f(c)}{h}$.

The analogous limit as $(h_1, h_2) \to (0, 0)$ almost always depends on the direction in which (h_1, h_2) approaches $(0, 0)$.

(And what do we divide by?)

Aside: there *is* still a way to define "the" derivative — but you won't see it in most calculus courses!

Partial derivatives

What we *can* do is choose the direction along which we let $(h_1, h_2) \to (0, 0)$.

- *Approaching parallel to the x axis*: this gives the partial derivative with respect to x. At a point (a, b) we define

$$f_x(a, b) = \lim_{h \to 0} \frac{f(a + h, b) - f(a, b)}{h}.$$

- *Approaching parallel to the x axis*: this gives us the partial derivative with respect to y. At a point (a, b) we define

$$f_y(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}.$$

Example

Let $f(x, y) = x^2y + 3x + 2$. Compute $f_x(1, 2)$:

1. Using the limit definition.
2. By realizing $f_x(1, 2) = g'(1)$, where $g(x) = f(x, 2)$.

More examples

Compute f_x and f_y, where:

1. $f(x, y) = x^2 y^3 + e^{2x+3y}$
2. $f(x, y) = \dfrac{x^2 \sin(y)}{x^2 + y^2}$
3. $f(x, y) = \sqrt{xy^3} + \tan(x^2 y)$

Second order derivatives

Just like in one variable, we can consider higher-order derivatives. But now, there's variety! Given $f(x, y)$, there are four possible second-order derivatives:

$$f_{xx}(x, y) = \frac{\partial}{\partial x} f_x(x, y) = \frac{\partial^2}{\partial x^2} f(x, y)$$
$$f_{xy}(x, y) = \frac{\partial}{\partial y} f_x(x, y) = \frac{\partial^2}{\partial y \partial x} f(x, y)$$
$$f_{yx}(x, y) = \frac{\partial}{\partial x} f_y(x, y) = \frac{\partial^2}{\partial x \partial y} f(x, y)$$
$$f_{yy}(x, y) = \frac{\partial}{\partial y} f_y(x, y) = \frac{\partial^2}{\partial y^2} f(x, y)$$

Clairaut's Theorem: if f has continuous second-order partial derivatives, then $f_{xy} = f_{yx}$.

Examples

Compute the second-order partial derivatives for $f(x, y) = x^4 \cos(x^3 e^y)$ and $g(x, y) = \ln(x^4 y^7)$.

Interpretation

What information are we computing when we find a partial derivative?

Tangent planes

We know that the curves $x = s, y = b, z = f(s, b)$ and $x = a, y = t, z = f(a, t)$ both lie on the surface $z = f(a, b)$. We also know that these curves intersect at the point $(a, b, f(a, b))$, and we know the slopes of their tangent lines. Those two lines lie in a common plane, called the *tangent plane*.

Theorem:

If $f(x, y)$ has continuous first-order partial derivatives at (a, b), then the surface $z = f(x, y)$ has a tangent plane approximation at $(a, b, f(a, b))$ given by

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b).$$

Example

Find the equation of the tangent plane to the surface $z = x^2ye^{x^2-y^2}$ at the point $(-1, 1, 1)$.