Math 2565, Spring 2020

The post-COVID lockdown power series wrap-up

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Overview

Warm-Up

Hi!

$\frac{d}{dx}(1-x)^{-1} = -(1-x)^{-2}(-1)$

1. Given the geometric series formula $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, determine a power series representation for $f(x) = \frac{x^4}{(1-x)^3}$.
2. Give an example of a power series whose interval of convergence is $(0, 6]$.

   $c = 3 \quad R = 3 \qquad \sum \frac{(-1)^n (x-3)^n}{n3^n}$
3. Use a power series to evaluate the integral $\int_0^2 e^{x^3}\, dx$

$$\sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}$$

$$\therefore \sum_{n=1}^{\infty} (n+1)(n)x^{n-1} = \frac{2}{(1-x)^3}$$

$$\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2} x^{n+4} = \frac{x^4}{(1-x)^3}$$

multiply by x^4.

$$\int_0^2 e^{x^3}\,dx$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \implies e^{x^3} = \sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{3n}}{n!}$$

$$\int e^{x^3}dx = \int\left(\sum_{n=0}^{\infty} \frac{x^{3n}}{n!}\right)dx = \sum_{n=0}^{\infty} \frac{1}{n!}\int x^{3n}\,dx$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\cdot\frac{x^{3n+1}}{3n+1} + C$$

$$\therefore \int_0^2 e^{x^3}dx = \sum_{n=0}^{\infty} \frac{2^{3n+1}}{n!\,(3n+1)}.$$

Taylor polynomials

Recall: given a function f differentiable at c, its *linear approximation* at c is given by

$f'(c)$

$$\ell(x) = f(c) + f'(c)(x - c).$$

Higher-order approximations are given by *Taylor polynomials*:

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k.$$

If a Taylor polynomial is centred at 0, we call it a *Maclaurin polynomial*.

Taylor series

The *Taylor series* of an infinitely differentiable function $f(x)$ is given by

$$f(x) = \underbrace{\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n}.$$

When c is zero, we get the Maclaurin series for f.

Equality of series and function

Recall the *Lagrange formula* for the remainder $R_n(x) = f(x) - P_n(x)$ obtained when we approximate a function f by its degree n Taylor polynomial:

$$R_n(x) = \frac{f^{(n+1)}(t)}{(n+1)!}(x-c)^{n+1},$$

where t is some number between x and c. For many common functions, it's not hard to show that $\lim_{n\to\infty} R_n(x) = 0$ for all x in the radius of convergence of the Taylor series. Letting $n \to \infty$ in the equality $\boxed{f(x) = P_n(x) + R_n(x)}$, we get

for any n

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n,$$

provided x is within the interval of convergence of the series, and is such that $R_n(x) \to 0$.

Common Taylor series

Give the Taylor series for:

1. $f(x) = e^x$, centred at 0.
2. $g(x) = \cos(x)$, centred at 0.
3. $h(x) = \ln(x)$, centred at 1.

Give the interval of convergence for each one.

$f'(x) = e^x, f''(x) = e^x, \ldots$

$f(0) = 1, f'(0) = 1, f''(0) = 1, \ldots$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$\cos(0) = 1$ $0 = 2(0)$

$-\sin(0) = 0$

$-\cos(0) = -1$ $2 = 2(1)$

$\sin(0) = 0$ $4 = 2(2)$

$\cos(0) = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$\ln(1) = 0$

$h'(x) = \frac{1}{x}$ $h'(1) = 1$

$h''(x) = \frac{-1}{x^2}$ $h''(1) = -1$

$h'''(x) = \frac{2}{x^3}$ $h'''(1) = 2$

$h^{(4)}(x) = -\frac{2(3)}{x^4}$ $h^{(4)}(1) = -2(3)$

$$\frac{h^{(n)}(1)}{n!} = \frac{(-1)^{n+1}(n-1)!}{n!} = \frac{(-1)^{n+1}}{n}$$

$$h(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n}$$

interval: $(0, 2]$

Manipulating power series

We can add, subtract, multiply and divide power series. Substitution works, too. (Sort of.) In the case of Taylor series, this is equivalent to doing the same to the corresponding functions.

Example

Determine Taylor series for:

1. $f(x) = \dfrac{x^3}{1-x} = x^3 \sum_{n=0}^{\infty} x^n = \sum_{n=0} x^{n+3}$
2. $g(x) = \dfrac{\sin(x)}{e^x} = \sin(x) e^{-x} = \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)}_{\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}} \left(1 - x + \frac{x^2}{2!} - \cdots\right)$
3. $h(x) = \sec(x)$
4. $s(x) = \sin(x^3) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{6n+3}}{(2n+1)!}$

$e^x = \sum \frac{x^n}{n!}$

$e^{-x} = \sum \frac{(-1)^n x^n}{n!}$

$\left(\sum a_n x^n\right)\left(\sum b_n x^n\right) = \sum c_n x^n$

$c_n = \sum_{k=0}^{n} a_k b_{n-k}$

$(x^3)^{2n+1} = x^{6n+3}$

$$\sec(x) = \frac{1}{\cos(x)}, \quad \text{and} \quad \cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

$$1 + \frac{x^2}{2!} + \frac{5}{4!}x^4 + \dots$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \enclose{longdiv}{1 - 0x^2 + 0x^4 + \dots}$$

$$-\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)$$

$$\frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$\frac{x^2}{2!} - \frac{x^4}{(2!)^2} + \frac{x^6}{2!4!} \dots$$

$$\frac{5x^4}{4!} - \dots$$

$$-\frac{1}{4!} + \frac{1}{2!2!}$$

$$= -\frac{1}{4\cdot 3\cdot 2!} + \frac{1}{2!2!}$$

$$= \frac{1}{2!}\left(\frac{1}{2} - \frac{1}{12}\right)$$

$$= \frac{1}{2!}\cdot\frac{5}{12} = \frac{5}{4!}$$

Binomial series

We have $(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$, for $|x| < 1$. But note the binomial coefficients are a bit different if α is not a positive integer:

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

$\binom{1/2}{0} = 1 \quad \binom{1/2}{1} = 1/2$

$\binom{1/2}{2} = \frac{1/2(-1/2)}{2!} = -1/8$

Example

Find Taylor series for:

1. $f(x) = \sqrt{1+2x} = (1+(2x))^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} (2x)^k$ α
2. $g(x) = \dfrac{1}{(1+x^2)^2} = \sum_{k=0}^{\infty} \binom{-2}{k} x^{2k}$ $\quad \binom{-2}{k}: 1, -2, 6, -24$ $\quad (-1)^k k!$

$= (1+x^2)^{-2}$

$(\alpha = -2)$

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^{n+1}(n+1)x^n$$

Creating new functions

Many new functions (not expressible in terms of elementary functions) arise as power series, often as solutions to differential equations.

Example (Bessel functions.)

The *Bessel functions* are named after Friedrich Bessel, who found them as solutions to Kepler's equations. They also show up in problems involving vibrations. The Bessel function of order zero is given by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

The Bessel function of order one is given by

$$J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}.$$

Limits using Taylor series

Example

Evaluate the limit:

1. $\lim_{x\to 0} \dfrac{\cos(x) - \left(1 - \frac{x^2}{2}\right)}{x^4} = \lim_{x\to 0} \dfrac{\left(\cancel{1} - \cancel{\frac{x^2}{2}} + \frac{x^4}{4!} - \cdots\right) - \left(\cancel{1} - \cancel{\frac{x^2}{2}}\right)}{x^4}$
2. $\lim_{x\to 0} \dfrac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2} = -\frac{1}{8}$

$= \lim_{x\to 0} \frac{1}{\cancel{x^4}}\left(\frac{\cancel{x^4}\,1}{4!} - \frac{x^{\cancel{6}\,2}}{6!} + \cdots\right) = \frac{1}{4!}$

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$$

$$\sqrt{1+x} - 1 - \frac{1}{2}x = -\frac{1}{8}x^2 + \cdots$$

Integrals using Taylor series

We saw earlier that given a power series $\sum_{n=0}^{\infty} a_x(x-c)^n$, we have

$\int_0^1 x^{4n}\,dx = \frac{x^{4n+1}}{4n+1}\Big|_0^1 = \frac{1}{4n+1}$

$$\int_a^b \left(\sum_{n=0}^{\infty} a_n(x-c)^n\right)\,dx = \sum_{n=0}^{\infty} a_n \int_a^b (x-c)^n\,dx.$$

Now that we know Taylor series replacements for many functions, we can use this to integrate.

Example

Compute the integral $\int_0^1 \frac{\sin(x^2)}{x^2}\,dx$. What is the error in your approxmiation if you use the first 5 terms in the sum to estimate the value of the integral?

$\frac{\sin(x^2)}{x^2} = \frac{1}{x^2}\left(\sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n+1)!}$

$\int_0^1 \frac{\sin(x^2)}{x^2}\,dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!\,(4n+1)}$

$a_6 = \frac{1}{13!(25)} = \frac{1}{155675520000}$

Examples by request

I'll try to fit in a few examples here based on what you asked for on the forum. If you've tuned in live you can suggest others in the comments.
Note: I have no questions planned on direction fields for the test.

A population model

$$\ln\left(\frac{P}{93-P}\right) = \frac{279}{500}t + \ln\left(\frac{48}{45}\right)$$

Consider the logistic population model $\dfrac{dP}{dt} = \dfrac{3}{500}\underline{P(93-P)}$.

1. For which values of P is the population increasing? Decreasing?
2. Find $P(t)$ given that $P(0) = 48$.

$-$ 0 $+$ 93 $-$ P'

increasing for $0 < P < 93$

decreasing for $P > 93$.

$$\underbrace{\frac{500}{P(93-P)}}_{\text{partial frac}}\, dP = 3\,dt$$

$$\frac{A}{P} + \frac{B}{93-P} = \frac{500}{P(93-P)}$$

$$A(93-P) + BP = 500$$

$$A = \frac{500}{93} \qquad B = \frac{500}{93}$$

$$\frac{500}{93}\left(\frac{1}{P} + \frac{1}{93-P}\right) = 3\,dt$$

$$\ln(P) - \ln(93-P) = \frac{279}{500}t + C$$

$t = 0$: $\ln(48) - \ln(45) = C$

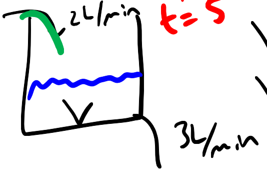
$$C = \ln\left(\frac{48}{45}\right)$$

A rate in/rate out problem

Suppose a water tank is being pumped out at 3 L / min . The water tank starts at 10 L of clean water. Water with toxic substance is flowing into the tank at 2 L / min , with concentration 20 g / L at time t. When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?

$t=5$

$X(0) = 0$



$V(0) = 10$

$V(t) = 10 - t$

$X(t)$: amount of substance

$$\mu(x) = e^{\int \frac{3}{10-t}\,dt} = e^{-3\ln(10-t)} = \frac{1}{(10-t)^3}$$

$$\frac{dx}{dt} = (20\text{g/L})(2\text{L/min}) - \left(\frac{X}{V}\right)(3\text{L/min}) = 40 - \frac{3X}{10-t}$$

$$\frac{dx}{dt} + \underbrace{\left(\frac{3}{10-t}\right)}_{P(t)} x = 40 \qquad \frac{d}{dt}\left(\frac{1}{(10-t)^3}x\right) = \frac{40}{(10-t)^3}$$

Convergence tests

Determine if the series converges:

1. $\displaystyle\sum_{n=0}^{\infty} \frac{n}{e^{n^2}}$
2. $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n + 3n}$
3. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+1}{\sqrt{n^3+4n+5}}$
4. $\displaystyle\sum_{n=1}^{\infty} \frac{7^k + k}{k! + 5}$

$$\int_0^\infty \frac{x}{e^{x^2}}\,dx = \int_0^\infty x e^{-x^2}\,dx = -\frac{1}{2} e^{-x^2}\Big|_0^\infty$$

$$\int x e^{-x^2}\,dx = -\frac{1}{2}\int e^u\,du = \frac{1}{2}$$

$u = -x^2 \quad du = -2x\,dx$

$$\frac{1}{2^n+3n} < \frac{1}{2^n} \qquad \sum \frac{1}{2^n} \text{ converges}$$

for large n $\approx \dfrac{n^2}{\sqrt{n^3}} = \boxed{n^{1/2}}$

$$= \sum_{n=1}^{\infty} \frac{7^k}{k!+5} + \sum_{n=1}^{\infty} \frac{k}{k!+5}$$

compare to $\dfrac{7^k}{k!}$ ratio test!

compare to $\dfrac{k}{k!} = \dfrac{1}{(k-1)!}$