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Parametric curves

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Overview

Warm-Up

Identify the curve defined by the parametric equations:

$$x = 2t - 1, y = -t + 4, t \in \mathbb{R}$$

$$x = \cos(2t), y = \sin(2t), t \in [0, \pi]$$

Parabolas

Defined as the locus (set) of points equidistant from a line (the *directrix*) and a point (the *focus*). If the vertex is at $(0,0)$, the focus is at $(0,p)$, and the directrix is $y=-p$, then the parabola has equation

$$y = \frac{1}{4p}x^2.$$

Ellipses

An *ellipse* is the set of all points P such that the sum of the distances from P to two points F_1, F_2, called the *foci* is constant. If the centre is at $(0,0)$, and the foci are at $(\pm c, 0)$, the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $c^2 = a^2 - b^2$. Vertices are at $(\pm a, 0)$ and $(0, \pm b)$.

Example

Sketch the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1.$

Hyperbolas

A hyperbola also has two foci. This time, we look for the set of all points where the *difference* of the distances between each point and the foci is constant. Result:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Example

Sketch the hyperbola $\dfrac{x^2}{9} - \dfrac{y^2}{4} = 1.$

Parametric equations

Usually we describe curves "explicitly", as graphs of functions: $y = f(X)$.

This is rather restrictive: lots of intesting curves are not graphs! (Like circles, ellipses, hyperbolas...)

One option is defining curves *implicitly* using equations of the form $f(x, y) = c$.

Another is to define them *parametrically*: we define x and y as functions of a third variable t.

An example you've seen: $x = \cos\theta$, $y = \sin\theta$.

From linear algebra: $\langle x, y\rangle = \langle 2, 3\rangle + t\langle 5, 1\rangle$

Examples of parametric curves

$$x = t^2 - t, y = 1 - t^2, t \in [-3, 3]$$

$$x = t^3 - t + 3, y = t^2 + 1, t \in [-2, 2]$$

$$x = \cos(t), y = \sin(2t), t \in [0, \pi]$$

Eliminating the parameter

Sometimes we can gain insight on a parametric curve by converting back to an equation relating x and y. Examples:

$$x = 3\cos(t) - 1, y = 2\sin(t) + 3, t \in [0, 2\pi].$$

$$x = \cos(t), y = \cos(2t), t \in [0, \pi]$$

$$x = \frac{1}{2+t}, y = \frac{3t+9}{3+t}$$

Note: plotting curves on a computer often relies on being able to parametrize them. But (except for graphs) this isn't always easy!

Smooth curves

A curve $x = f(t), y = g(t)$ is considered *smooth* if f and g are both differentiable, and $f'(t)$ and $g'(t)$ are never simultaneously zero. Some people will also require that a smooth curve has no *self-intersections*. Finding points of self-intersection is an algebraic nightmare.

Example

1. Find the points where $x = t^2 - 4t, y = t^3 - 2t^2 - 4t$ is not smooth.
2. Find the points where $x = \cos(t), y = 2\cos(t)$ is not smooth.

Tangent lines

Our definition of *smooth curve* is natural in one sense: it means we can find tangent lines! Slope of the tangent is given by $\frac{dy}{dx}$. Chain rule gives:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

so

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}.$$

If either $x'(t)$ or $y'(t)$ is undefined, we can't compute the slope of the tangent. If both are zero, the slope is indeterminate. (If $x'(t) = 0$ but $y'(t) \neq 0$, we have a vertical tangent!) (Aside:) a vector in the direction of the tangent line at $(x(t_0), y(t_0))$ is $\langle x'(t_0), y'(t_0)\rangle$.

Examples

Find the equations of the tangent lines as indicated:

$x = t^2 - 1, y = t^3 - t$, at $t = 0$ and $t = 1$

$x = \tan(t), y = \sec(t)$, at $t = \pi/4$.

Area

We can compute area as usual. For area under a curve: $A = \int_a^b f(x)\,dx$. If $x = g(t), y = f(g(t))$, get $A = \int_{t_0}^{t_1} f(g(t))g'(t)\,dt$. This still makes sense for a general parametric curve if we're careful about interpretation. For a *closed* curve $x = g(t), y = h(t)$ with *counterclockwise orientation*, the area enclosed is

$$A = -\int_{t_0}^{t_1} h(t)g'(t)\,dt.$$

Examples

Find the area encolosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Find the area encolosed by the "teardrop" in the curve $x = t^3 - t, y = t^2 - 1$.