Math 2565, Spring 2020

Parametric curves

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Overview

1 Warm-Up

2 Conic Sections

3 Parametric Curves

4 Calculus with parametric curves

Warm-Up

Today: 10.1 (briefly), 10.2, 10.3

Identify the curve defined by the parametric equations:

$$x = 2t - 1, y = -t + 4, t \in \mathbb{R} \longrightarrow t = 4 - y \Rightarrow x = 2(4-y) - 1$$

$$= 8 - 2y - 1$$

$$= 7 - 2y$$

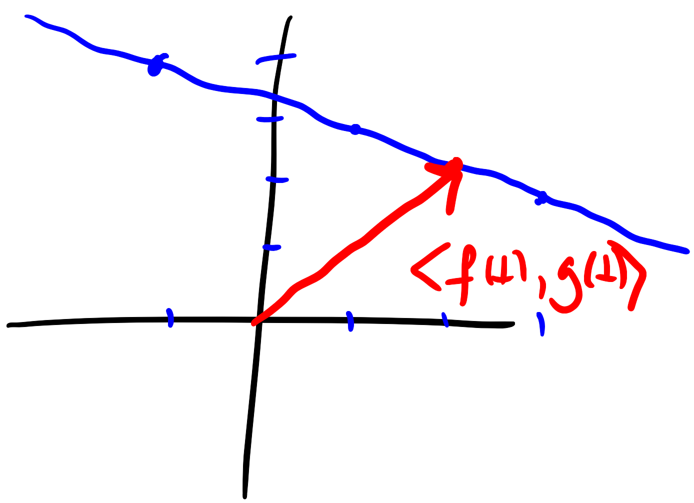
$$x = 7 - 2y$$

$$y = -\frac{1}{2}x + \frac{7}{2}$$

$$x = \cos(2t), y = \sin(2t), t \in [0, \pi]$$

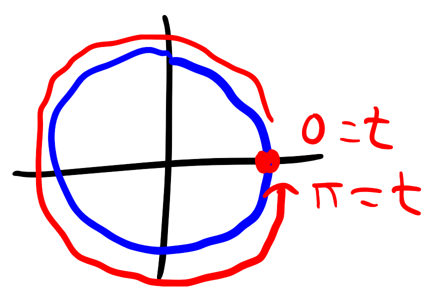
$$x = f(t)$$
$$y = g(t)$$

t	$2t-1$	$-t+4$
0	-1	4
1	1	+3
2	3	+2



$$\cos^2(2t) + \sin^2(2t) = 1$$

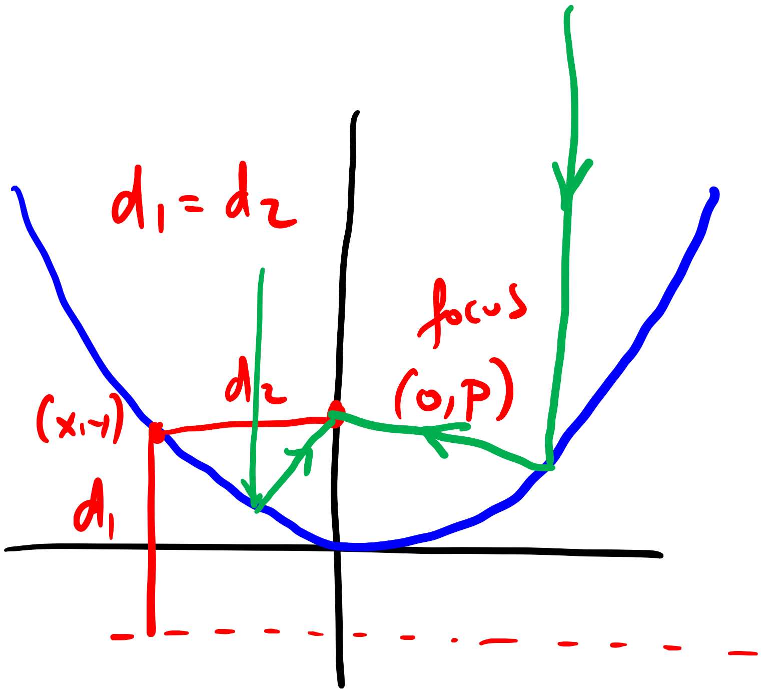
$$\boxed{x^2 + y^2 = 1}$$



Parabolas

Defined as the locus (set) of points equidistant from a line (the *directrix*) and a point (the *focus*). If the vertex is at $(0,0)$, the focus is at $(0,p)$, and the directrix is $y=-p$, then the parabola has equation

$$y = \frac{1}{4p}x^2.$$



$y = -p$

Ellipses

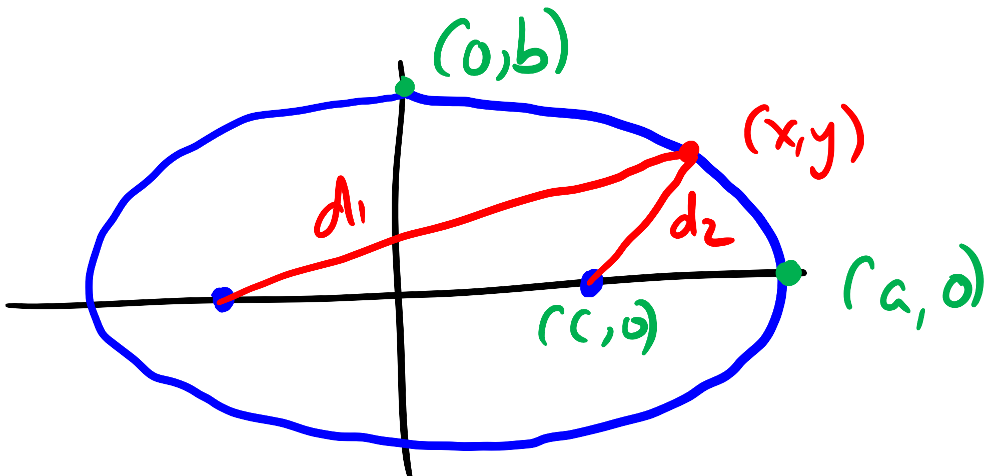
An *ellipse* is the set of all points P such that the sum of the distances from P to two points F_1, F_2, called the *foci* is constant. If the centre is at $(0,0)$, and the foci are at $(\pm c, 0)$, the ellipse has equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Shifted:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

where $c^2 = a^2 - b^2$. Vertices are at $(\pm a, 0)$ and $(0, \pm b)$.



$d_1 + d_2 = \text{constant}.$

$a > b$

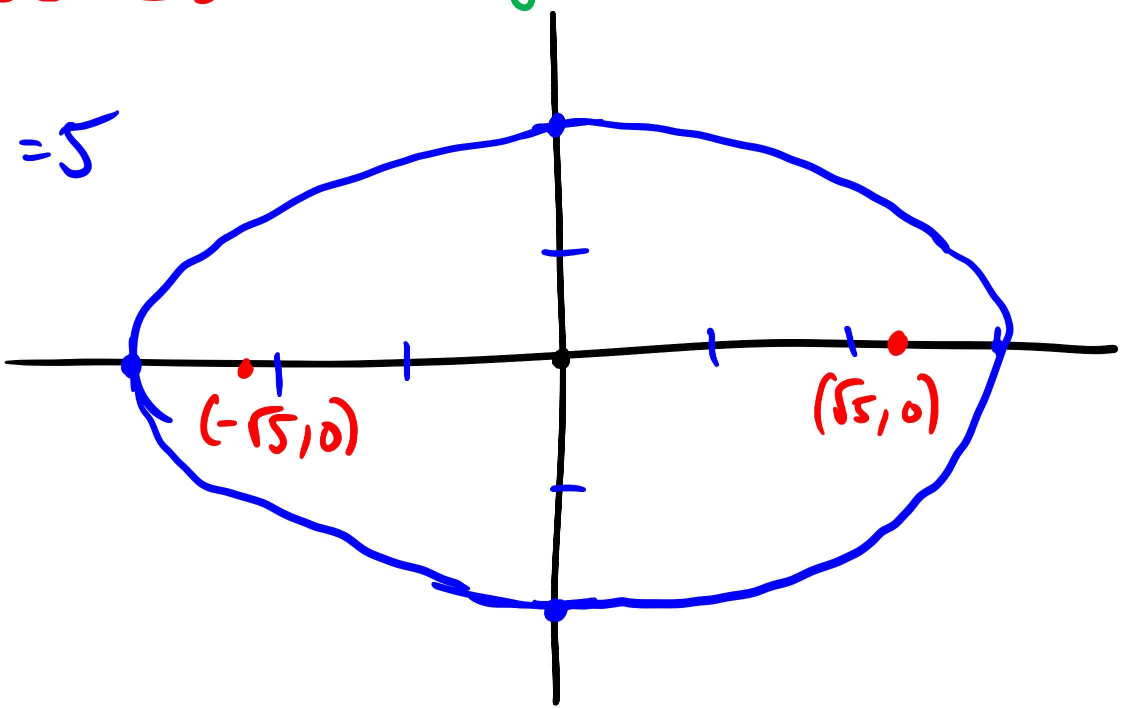
Example

Sketch the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$.

$x^2 \leq 9$ $\qquad$ $y^2 \leq 4$

$-3 \leq x \leq 3$ $\qquad$ $-2 \leq y \leq 2$

$c^2 = 9 - 4 = 5$



Hyperbolas

A hyperbola also has two foci. This time, we look for the set of all points where the *difference* of the distances between each point and the foci is constant. Result:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{or} \quad -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

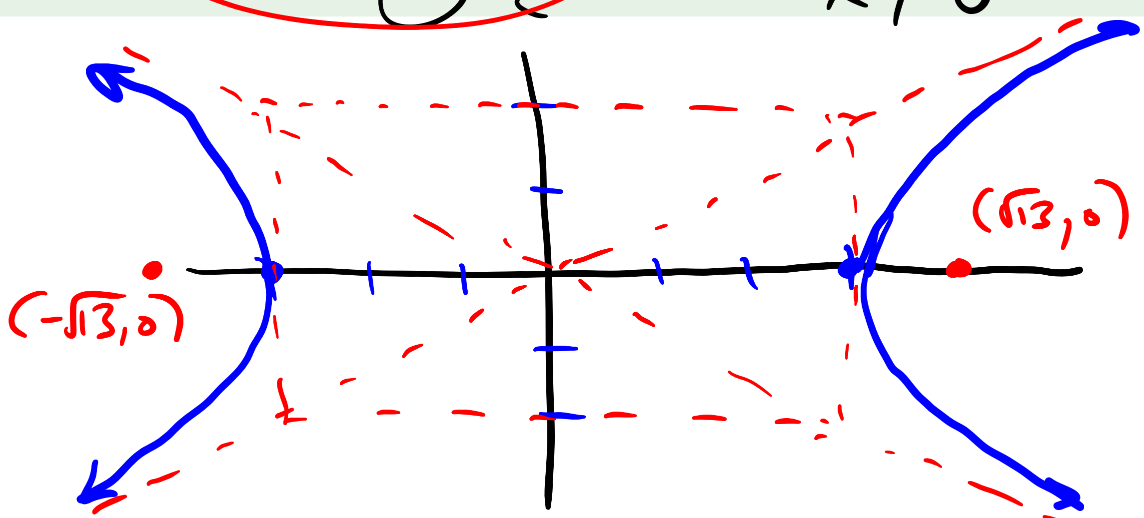
$\frac{y^2}{4} = \frac{x^2}{9} - 1 \qquad y^2 = \frac{4}{9}\sqrt{x^2-9} \;. \quad y = \pm\frac{2}{3}\sqrt{x^2-9} \approx \pm\frac{2}{3}x$

Example

Sketch the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1.$ 2^2

$y = 0: \quad x^2 = 9 \quad x = \pm 3$

$x \neq 0$



Focus:

$c^2 = a^2 + b^2$

$c^2 = 13$

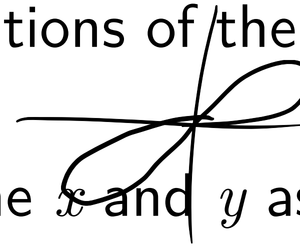
Parametric equations

Usually we describe curves "explicitly", as graphs of functions: $y = f(X)$.

This is rather restrictive: lots of intesting curves are not graphs! (Like circles, ellipses, hyperbolas...)

One option is defining curves *implicitly* using equations of the form $f(x, y) = c$.

Eg: $(x^2+y^2)^2 = 4xy$



Another is to define them *parametrically*: we define x and y as functions of a third variable t.

An example you've seen: $x = \cos\theta$, $y = \sin\theta$.

From linear algebra: $\underbrace{\langle x, y\rangle = \langle 2, 3\rangle + t\langle 5, 1\rangle}$

$x = 2 + 5t$

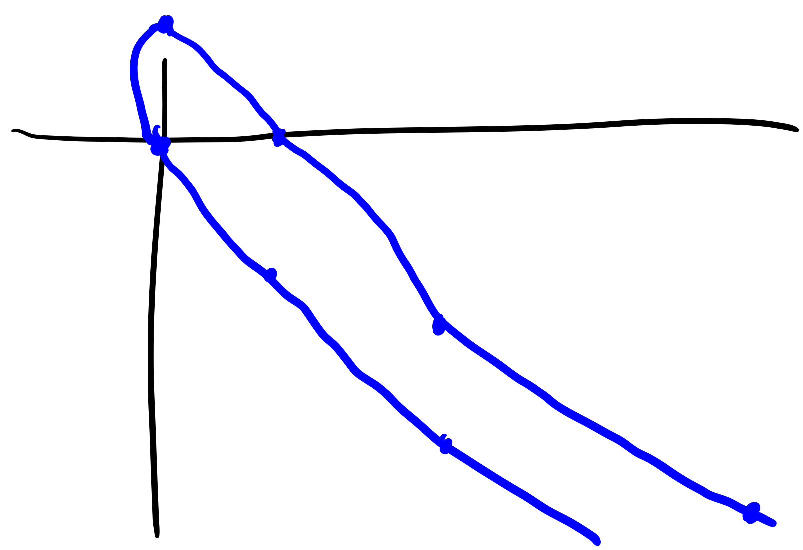
$y = 3 + t$

Examples of parametric curves

1. $x = t^2 - t, y = 1 - t^2, t \in [-3, 3]$
2. $x = t^3 - t + 3, y = t^2 + 1, t \in [-2, 2]$
3. $x = \cos(t), y = \sin(2t), t \in [0, \pi]$

1.

t	x	y
-3	12	-8
-2	6	-3
-1	2	0
0	0	1
1	0	0
2	2	-3
3	6	-8



Eliminating the parameter

Sometimes we can gain insight on a parametric curve by converting back to an equation relating x and y. Examples:

$$x = 3\cos(t) - 1, y = 2\sin(t) + 3, t \in [0, 2\pi].$$

$$x = \cos(t), y = \cos(2t), t \in [0, \pi]$$

$$x = \frac{1}{2+t}, y = \frac{3t+9}{3+t}$$

Note: plotting curves on a computer often relies on being able to parametrize them. But (except for graphs) this isn't always easy!

Smooth curves

$f'(t)$

A curve $x = f(t), y = g(t)$ is considered *smooth* if f and g are both differentiable, and $f'(t)$ and $g'(t)$ are never simultaneously zero. Some people will also require that a smooth curve has no *self-intersections.* Finding points of self-intersection is an algebraic nightmare.

Example

1. Find the points where $x = t^2 - 4t, y = t^3 - 2t^2 - 4t$ is not smooth.
2. Find the points where $x = \cos(t), y = 2\cos(t)$ is not smooth.

$x = t^2 - 4t$

$x'(t) = 2t - 4$

$x'(2) = 0$

$x(2) = -4$

$y(2) = -8$

$y = t^3 - 2t^2 - 4t$

$y'(t) = 3t^2 - 4t - 4$

$y'(2) = 3(4) - 4(2) - 4 = 0$

2. $y = 2x$

Tangent lines

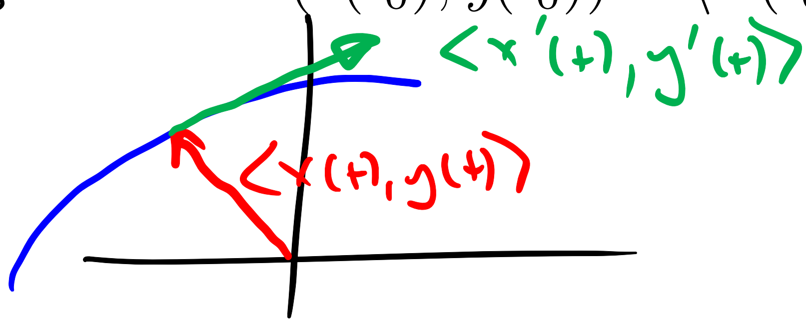
Our definition of *smooth curve* is natural in one sense: it means we can find tangent lines! Slope of the tangent is given by $\dfrac{dy}{dx}$. Chain rule gives:

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt},$$

so

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)}. \quad = \frac{y'(t)}{x'(t)}$$

If either $x'(t)$ or $y'(t)$ is undefined, we can't compute the slope of the tangent. If both are zero, the slope is indeterminate. (If $x'(t) = 0$ but $y'(t) \neq 0$, we have a vertical tangent!) (Aside:) a vector in the direction of the tangent line at $(x(t_0), y(t_0))$ is $\langle x'(t_0), y'(t_0)\rangle$.



Examples

Find the equations of the tangent lines as indicated:

1 $x = t^2 - 1$, $y = \underline{t^3 - t}$, at $t = 0$ and $t = 1$

2 $x = \tan(t)$, $y = \sec(t)$, at $t = \pi/4$.

1. $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{3t^2 - 1}{2t}$

$t = 0$: vertical tangent

tangent: $x = -1$

$(x(0), y(0)) = (-1, 0)$

$(x(1), y(1)) = (0, 0)$

$t = 1$: $\dfrac{dy}{dx} = \dfrac{2}{2} = 1$

tangent: $y = x$

2. $\dfrac{dy}{dx} = \dfrac{y'(t)}{x'(t)} = \dfrac{\sec(t)\tan(t)}{\sec^2(t)} = \dfrac{\tan(t)}{\sec(t)} = \sin(t)$

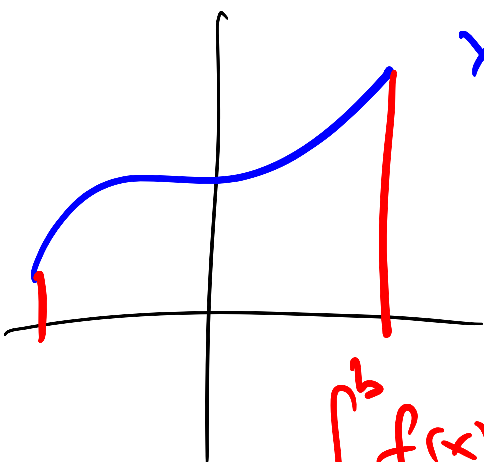
at $\frac{\pi}{4}$: $x = 1$, $y = \sqrt{2}$, $m = 1/\sqrt{2}$ tangent: $y = \sqrt{2} + \frac{1}{\sqrt{2}}(x - 1)$

Area

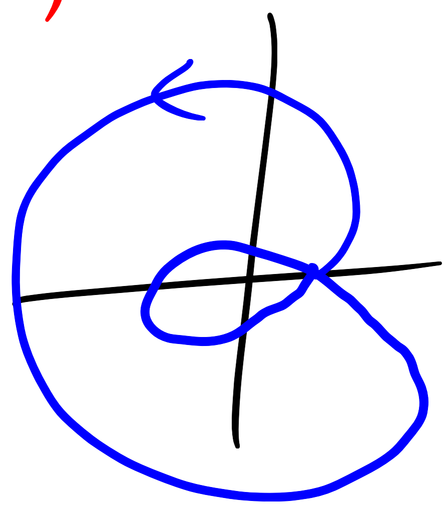
We can compute area as usual. For area under a curve: $A = \int_a^b f(x)\,dx$. If $x = g(t)$, $y = f(g(t))$, get $\boxed{A = \int_{t_0}^{t_1} f(g(t))g'(t)\,dt.}$ This still makes sense for a general parametric curve if we're careful about interpretation. For a *closed* curve $x = g(t), y = h(t)$ with *counterclockwise orientation*, the area enclosed is

$$\boxed{A = -\int_{t_0}^{t_1} h(t)g'(t)\,dt.}$$

$x = t,\ y = f(t) \quad (y = f(x))$



$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$$



Examples

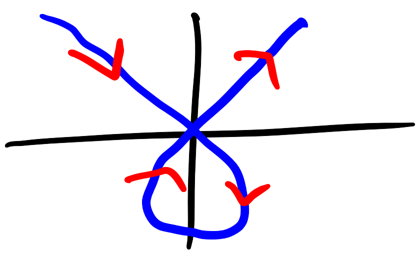
1. Find the area encolosed by the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1.$
2. Find the area encolosed by the "teardrop" in the curve $x = t^3 - t, y = t^2 - 1.$

1. $x = a\cos(t)$ $y = b\sin(t)$, $t \in [0, 2\pi]$

$$A = -\int_0^{2\pi} y(t)x'(t)\,dt = -\int_0^{2\pi} b\sin(t)(-a\sin(t))\,dt$$

$$= ab\int_0^{2\pi} \sin^2(t)\,dt = ab\int_0^{2\pi}\left(\frac{1-\cos(2t)}{2}\right)dt$$

$$= \pi ab.$$

2. 

$$A = \int_{-1}^{1} y(t)x'(t)\,dt$$