Math 2565, Spring 2020

Polar Coordinates

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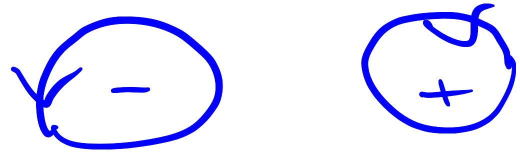
Overview

Warm-Up



Find the area encolosed by the astroid $x = \cos^3(t), y = \sin^3(t), t \in [0, 2\pi]$. How about its length?

$$A = \pm \int_a^b y(t) \underbrace{x'(t)\,dt}_{dx}$$

$$y(t) = \sin^3(t)$$

$$x'(t) = -3\cos^2(t)\sin(t)$$

$$A = -\int_0^{2\pi} -3\sin^4(t)\cos^2(t)\,dt$$

$$= +\int_0^{2\pi} 3\left(\frac{1-\cos(2t)}{2}\right)^2\left(\frac{1+\cos(2t)}{2}\right)dt$$

$$= +\frac{3}{8}\int_0^{2\pi}\left(\frac{3}{2} - 2\cos(2t) + \frac{\cos(4t)}{2}\right)(1+\cos(2t))\,dt$$

$$= +\frac{3}{8}\int_0^{2\pi}\left(\frac{3}{2} - \frac{1}{2}\cos(2t) - 2\cos^2(2t) + \frac{\cos(4t)}{2} + \frac{\cos(4t)\cos(2t)}{2}\right)dt$$

$$-1 - \cos(4t)$$

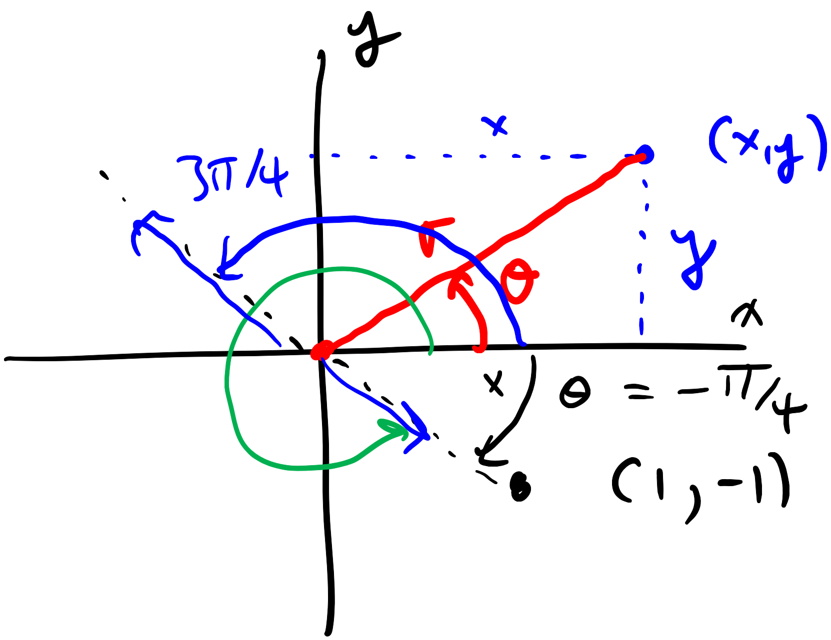
$$= \frac{3}{16}(2\pi)$$

$$= \frac{3\pi}{8}.$$

The polar coordinate system

Polar coordinates are defined in terms of the usual Cartesian plane. But instead of using x and y to locate a point, we use r and θ.

- r: distance from the origin to the point. (Defines a circle)
- θ: direction of the point. (Defines a point on the circle)

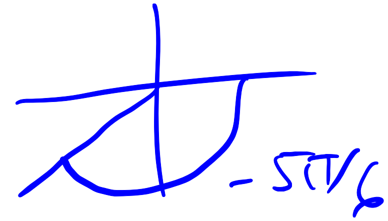


$$r^2 = x^2 + y^2 \qquad x = r\cos\theta$$
$$\tan\theta = y/x \qquad y = r\sin\theta$$

$\theta = -\pi/4$

$$(1,-1) \rightsquigarrow (\sqrt{2}, -\pi/4) = (\sqrt{2}, 7\pi/4)$$
$$= (-\sqrt{2}, 3\pi/4)$$

Converting

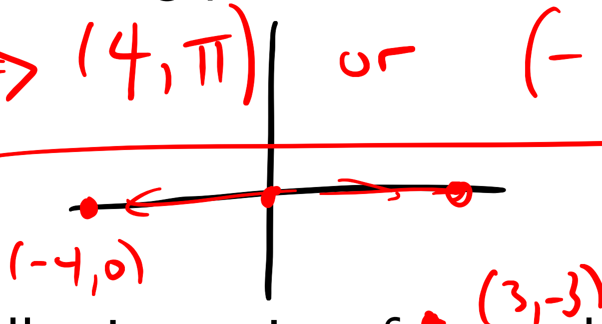


Example

Convert the following points from rectangular to polar coordinates:

$r = 0$ $\quad$ r $\quad$ θ

1. $(-4, 0)$ $\Rightarrow (4, \pi)$ or $(-4, 0)$
2. $(3, -3)$ $\rightarrow$ $r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$, $\theta = -\pi/4$
3. $(0, 0)$



Convert the following points from polar to rectangular coordinates:

1. $(3, \pi/3)$ $\quad x = 3\cos(\pi/3),\ y = 3\sin(\pi/3) \rightarrow (3/2, 3\sqrt{3}/2)$
2. $(-2, \pi/4)$ $\quad x = -2\cos(\pi/4),\ y = -2\sin(\pi/4) \rightarrow (-\sqrt{2}, -\sqrt{2})$
3. $(1, -5\pi/6)$ $\quad x = 1\cos(-5\pi/6) = -\frac{\sqrt{3}}{2}$ $\quad y = 1\sin(-5\pi/6) = -\frac{1}{2}$.

$x = r\cos\theta$

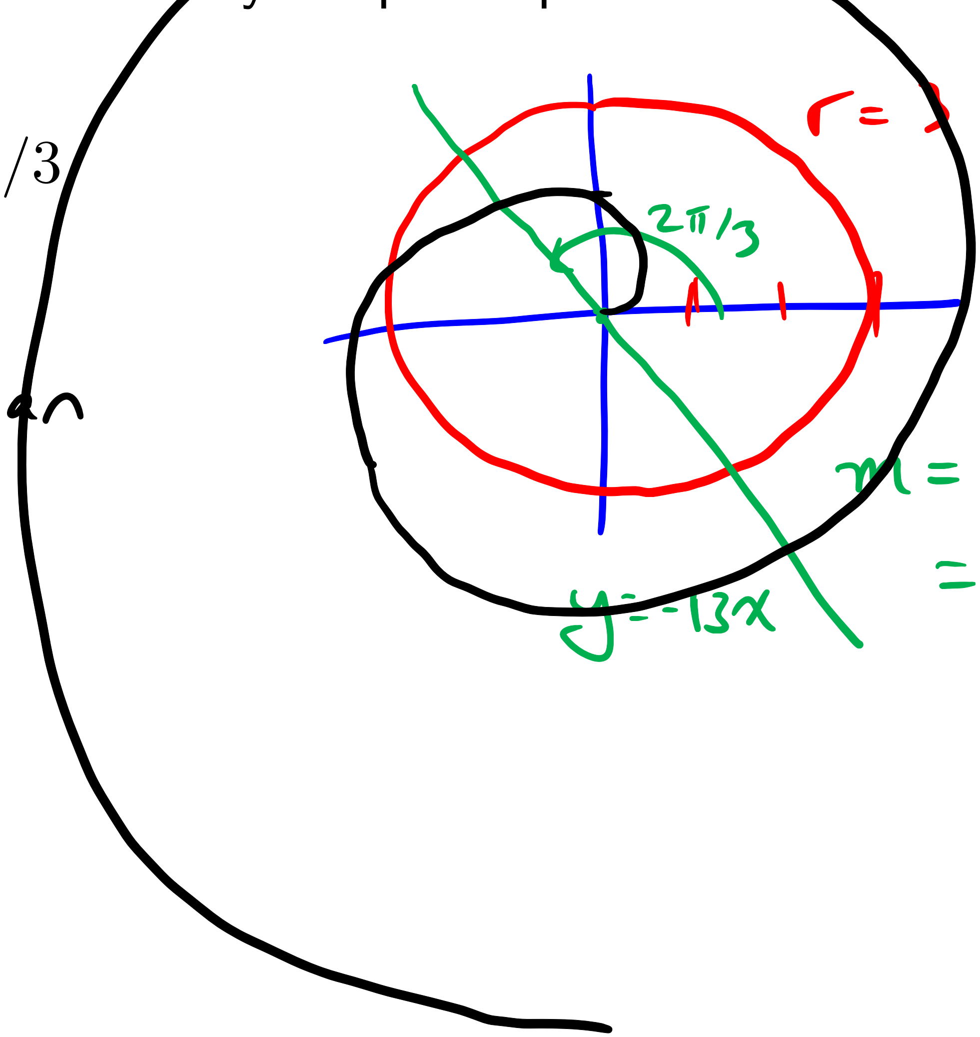
$y = r\sin\theta$

Simple polar curves

Some curves are really simple in polar coordinates. For example:

1. $r = 3$
2. $\theta = 2\pi/3$
3. $r = \theta$

Archimedean spiral



$(x^2 + y^2 = 9)$

$m = \tan\left(\frac{2\pi}{3}\right)$

$= -\sqrt{3}$

Simple rectangular curves

Some curves are simple in rectangular coordinates but not so great in polar:

1. $y = 3x + 2$ ✓. $r\sin\theta = 3r\cos\theta + 2$
   $r(\sin\theta - 3\cos\theta) = 2$
   $\therefore r = \dfrac{2}{\sin\theta - 3\cos\theta}$
2. $y = 3$
3. $y = x^2$

2. $r\sin\theta = 3 \implies r = 3\csc\theta$

3. $r\sin\theta = r^2\cos^2\theta \implies r = 0$, or

$\dfrac{\sin\theta}{\cos^2\theta} = r = \sec\theta\tan\theta$.

$r(0) = 0$

we didn't lose this!

Polar curves: limaçons

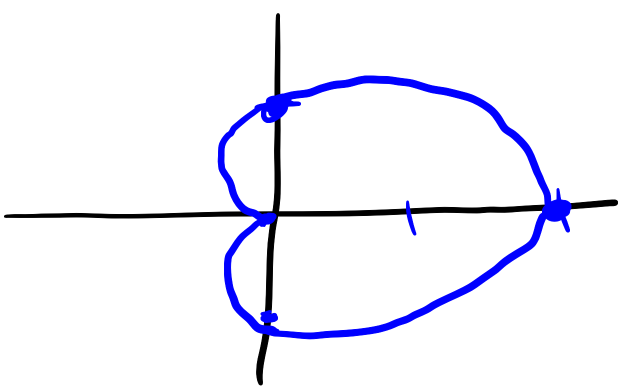
A typical polar curve has the form $r = f(\theta)$. We can generate a lot of interesting curves this way.

Example

Describe (sketch) the polar curve:

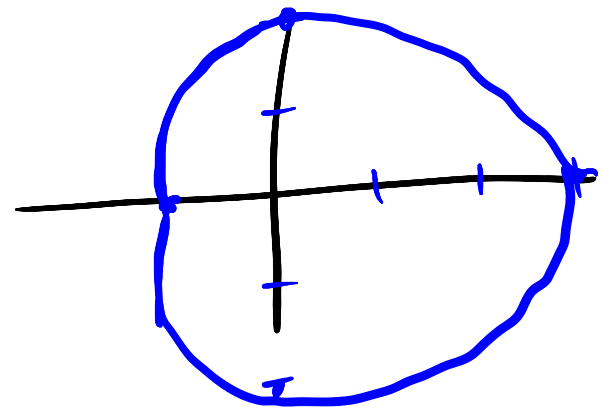
1. $r = 1 + \cos\theta$ (a = b)
2. $r = 2 + \cos\theta$ (a > b)
3. $r = 1 + 2\sin\theta$ (a < b)

1.

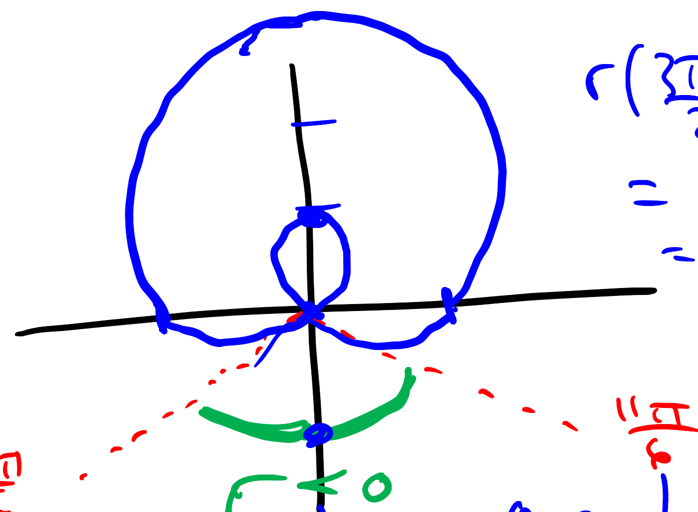


$r = 1 + \cos\theta$

2.



3.



$r\left(\frac{3\pi}{2}\right) = 1 - 2 = -1$

$r = 1 + 2\sin\theta = 0 \Rightarrow \sin\theta = -\frac{1}{2}$

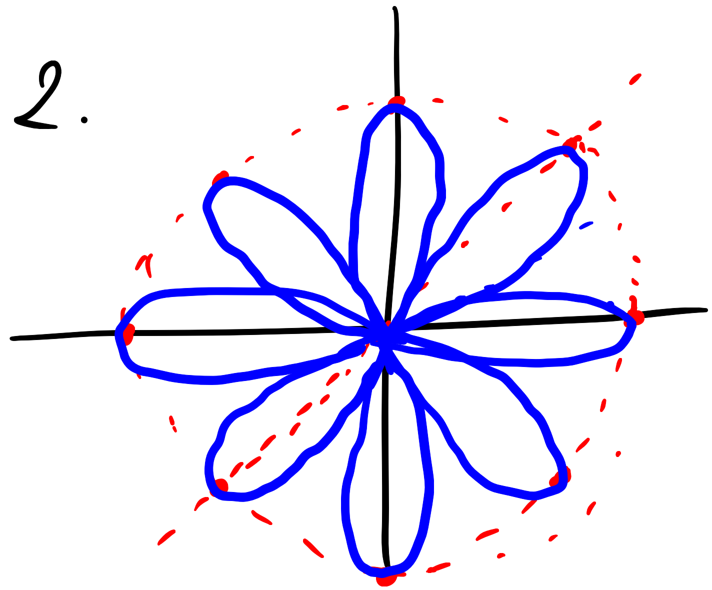
$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$

Polar curves: roses

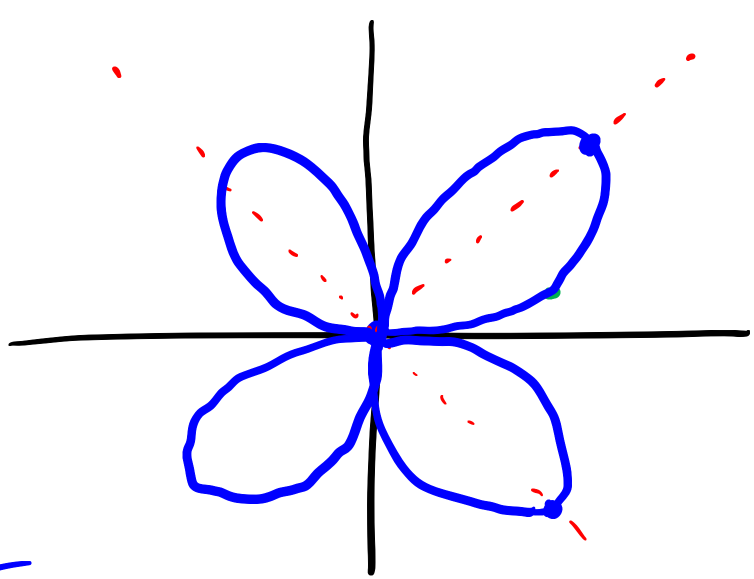
Describe (sketch) the polar curve:

1. $r = \sin(2\theta)$
2. $r = \cos(4\theta)$

2.



1.



θ	r
0	0
$\pi/4$	1
$\pi/2$	0

$\leftarrow \theta = \pi/12 \Rightarrow r = 1/2$

$3\pi/4: r = -1$ etc

$r = f(\theta)$

$x = r\cos\theta = f(\theta)\cos\theta$

$y = r\sin\theta = f(\theta)\sin\theta.$

Tangent lines

Polar curves are really parametric curves: $x = r\cos\theta$ and $y = r\sin\theta$, so $r = f(\theta)$ defines $x(\theta), y(\theta)$. That gives $\dfrac{dy}{dx} = \dfrac{y'(\theta)}{x'(\theta)}$.

Example

Find the equation of the tangent line to $r = \cos(3\theta)$ when $\theta = \pi/4$.

$y = 1/2.$

$$x = \cos(3\theta)\cos\theta$$
$$y = \sin(3\theta)\sin\theta$$

$$y(\pi/4) = \sin\left(\frac{3\pi}{4}\right)\sin\frac{\pi}{4} = \frac{1}{2}$$

$$\frac{dy}{dx} = \frac{y'(\theta)}{x'(\theta)} = \frac{3\cos(3\theta)\sin\theta + \sin(3\theta)\cos\theta}{-3\sin(3\theta)\cos\theta - \cos(3\theta)\sin\theta}$$

at $\pi/4$: $m = \dfrac{-3/\sqrt{2}\cdot 1/\sqrt{2} + 3/\sqrt{2}\cdot 1/\sqrt{2}}{\ldots} \leftarrow 0$

Example

1. Determine the location of the horizontal and vertical tangents of the limaçons $r = 1 + 2\cos\theta$.
2. Find the equations of the tangent lines to the curve at the origin. (There are two!)

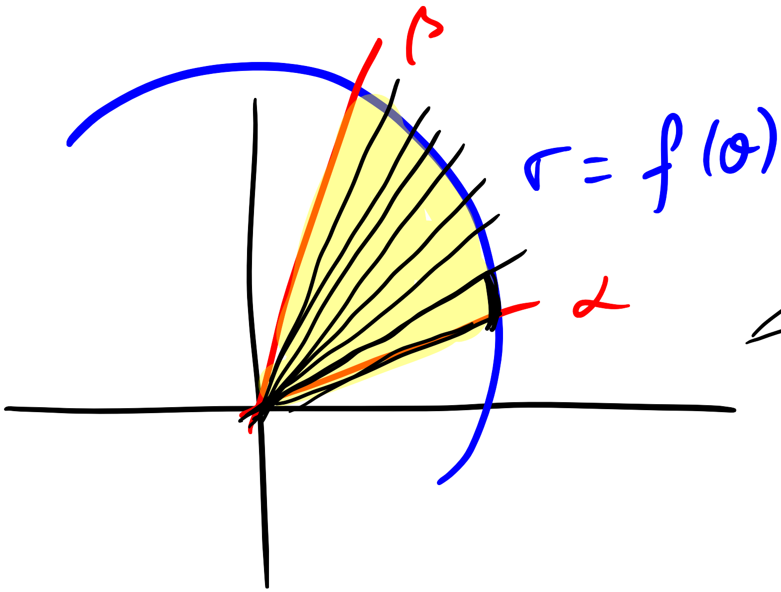
$$y(\theta) = (1+2\cos\theta)\sin\theta = \sin\theta + 2\sin\theta\cos\theta$$
$$= \sin\theta + \sin(2\theta)$$

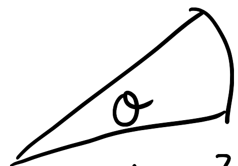
$$y'(\theta) = \cos\theta + 2\cos(2\theta)$$
$$= \cos\theta + 2(2\cos^2\theta - 1)$$
$$= 4\cos^2\theta + \cos\theta - 2$$

$\hookrightarrow$ use quadratic formula to find $\cos\theta$.

Area

The polar area formula is given by $A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$

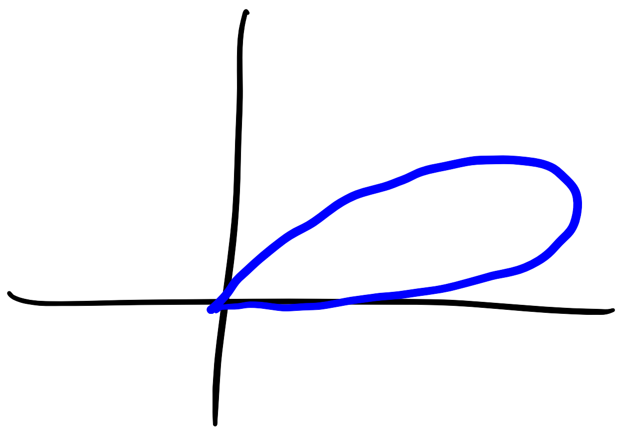




$\frac{1}{2} r^2 \theta$

Example

Find the area enclosed by one leaf of the curve $r = \sin(3\theta)$.



$0 \leq \theta \leq \frac{\pi}{3}$

$r = 0$:

$\sin(3\theta) = 0$

$3\theta = k\pi$

$\theta = \frac{k\pi}{3}$

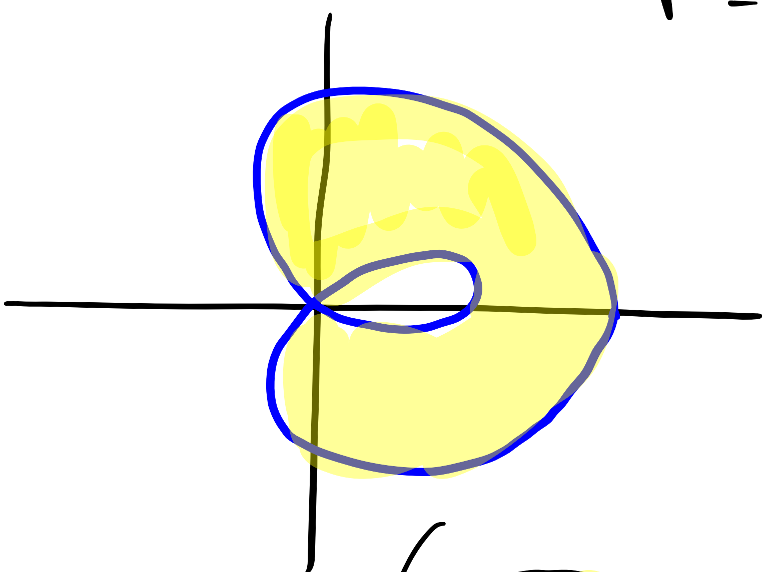
$$A = \frac{1}{2}\int_0^{\pi/3} \sin^2(3\theta)\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/3} \left(\frac{1-\cos(6\theta)}{2}\right) d\theta$$

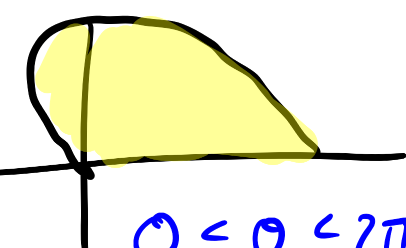
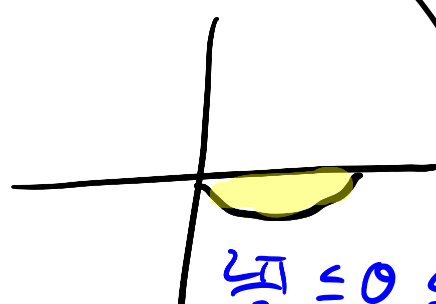
Example

Find the area that lies inside the outer loop of the limaçon $r = 1 + 2\cos\theta$, but outside the inner loop.

$r = 0$: $\cos\theta = -1/2 \Rightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$



$$A = \int_0^{2\pi/3} (1+2\cos\theta)^2\,d\theta - \int_{2\pi/3}^{\pi} (1+2\cos\theta)^2\,d\theta$$

Area: ( $-$ ) $\times 2$

$0 \leq \theta \leq 2\pi/3$ $\qquad$ $\frac{2\pi}{3} \leq \theta \leq \pi$

Example

Find the area that lies inside the curve $r = 4\cos\theta$ but outside the curve $r = 4\sin\theta$

Arc length

(If we have time.)