Math 2565, Spring 2020

Functions of several variables

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Overview

Warm-Up

Find the area that lies inside the outer loop of the limaçon $r = 1 + 2\cos\theta$, but outside the inner loop. Find the length of the inner loop for the above limaçon. Find the area that lies inside the curve $r = 4\cos\theta$ but outside the curve $r = 4\sin\theta$

Cartesian coordinates in 3D

Lines and planes

Lines in $\mathbb{R}^3$ are definited using parametric equations:

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

Planes are defined by a single linear relation: $ax + by + cz = d$. If you've done Math 1410, you've seen how to describe these using vectors.

Cylinders

A *cylinder* is a surface where the *traces* (cross-sections) parallel to a given plane are all the same curve. Typically these come from an equation involving only two of three variables. For example:

$$y = x^2 \quad z = e^y \quad z = \sin(x + y)$$

Quadric surfaces

These are the 3D analogues of the conic sections. Each one is defined by a quadratic equation in three variables.

Example

Describe the surface

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$

Traces

Traces are curves that lie on surfaces. These curves are formed by intersecting a surface with a plane. Usually we consider traces with respect to planes parallel to the coordinate planes. These are the easiest to find, and are key to helping us visualize a surface.

Ellipsoids

Elliptic paraboloids

Hyperbolic paraboloids

Maintaining with tradition, we will now watch Sean try (and fail) to sketch a hyperbolic paraboloid.

Hyperboloids

Cones

Surfaces in general

More general surfaces in $\mathbb{R}^3$ can be obtained in several ways. Three common ones:

As a *graph* $z = f(x, y)$

As a *level surface* $f(x, y, z) = k$

As a *parametric surface* $x = f(u, v), y = g(u, v), z = h(u, v)$.

In Math 2565 we look mostly at the first case, and not at all at the third.

Functions of two variables

A function f of two variables takes a point $(x, y) \in \mathbb{R}^2$ as an input, and gives a point $z = f(x, y)$ as an output. If $D \subseteq \mathbb{R}^2$ is the domain of f we might write $f : D \to \mathbb{R}$ to emphasise the types of input and output.
Examples:

Examples

For the following functions, compute the values at $(0,0),(1,2)$, and $(3,-2)$. Determine the domain and range.

$$f(x,y) = x^2 - y^2$$

$$f(x,y) = e^{-(x^2+y^2)}$$

$$f(x,y) = \ln(xy)$$

Graphing functions of two variables

How do we even begin to visualize a function of two variables? (Or three, or four, or...) The key is to consider traces of the graph $z = f(x, y)$. Traces in planes $z = k$ are especially useful. These are called *level curves*. Examples:

$$z = x - y^2 \quad z = 4x^2 + y^2 \quad z = \sin(x)\cos(y)$$

Level curves vs. contour plots

You'll see mentions of both level curves and contour plots. These are related, but different. A level curve lies *on the surface*, in a plane $z = k$. A contour curve is the curve we get in the xy plane if we set $z = 0$. A contour plot is a collection of contour curves $f(x, y) = k$ for different values of k.