Math 2565, Spring 2020

Functions of several variables

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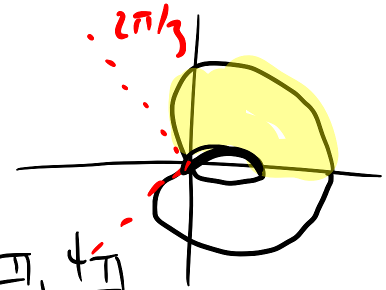
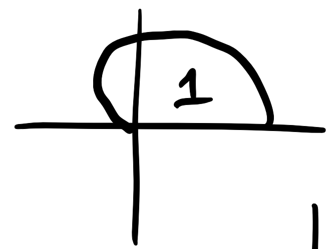
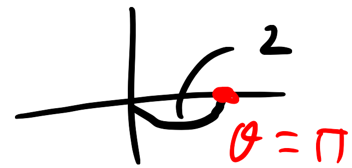
Overview

Warm-Up

Find the area that lies inside the outer loop of the limaçon $r = 1 + 2\cos\theta$, but outside the inner loop. Find the length of the inner loop for the above limaçon. Find the area that lies inside the curve $r = 4\cos\theta$ but outside the curve $r = 4\sin\theta$







1. $r = 1 + 2\cos\theta$

$r = 0$: $\cos\theta = -1/2$ $\quad \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$A_1 = \frac{1}{2}\int_0^{2\pi/3} (1+2\cos\theta)^2 \, d\theta$$

$$A_2 = \frac{1}{2}\int_{2\pi/3}^{\pi} (1+2\cos\theta)^2 \, d\theta$$

$$A = 2A_1 - 2A_2$$

$$(1+2\cos\theta)^2 = 1 + 4\cos\theta + \underline{4\cos^2\theta}$$
$$= 1 + 4\cos\theta + 2(1+\cos 2\theta)$$
$$= 3 + 4\cos\theta + 2\cos 2\theta$$

Arc length

$x = f(t) \qquad y = g(t)$

$$ds^2 = dx^2 + dy^2 \qquad dx = f'(t)\,dt \qquad dy = g'(t)\,dt$$

$$ds = \sqrt{f'(t)^2 + g'(t)^2}\,dt \qquad \text{(parametric)}$$

Polar: $r = f(\theta) \qquad x = f(\theta)\cos\theta \qquad y = f(\theta)\sin\theta$

$$dx = \left(f'(\theta)\cos\theta - f(\theta)\sin\theta\right)d\theta$$

$$dy = \left(f'(\theta)\sin\theta + f(\theta)\cos\theta\right)d\theta$$

$$ds = \sqrt{dx^2 + dy^2} = \left((f'(\theta))^2 + f(\theta)^2\right)d\theta$$

For $r = 1 + 2\cos\theta = f(\theta)$

$$\frac{dr}{d\theta} = -2\sin\theta = f'(\theta)$$

$$f(\theta)^2 + f'(\theta)^2 = 1 + 4\cos\theta + 4\cos^2\theta + 4\sin^2\theta$$

$$= 5 + 4\cos\theta.$$

$$\boxed{L = \int_{2\pi/3}^{4\pi/3} \sqrt{5 + 4\cos\theta}\,d\theta} \longleftarrow \text{use numerical}$$

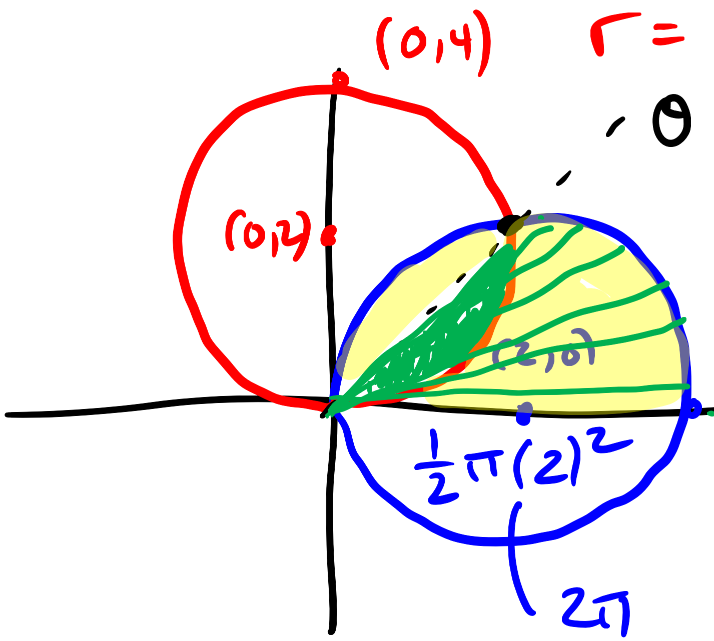
$r = 4\sin\theta \implies r^2 = 4r\sin\theta$

$x^2 + y^2 = 4y$

$x^2 + (y-2)^2 = 4$

$\underbrace{r = 4\cos\theta}_{(x-2)^2 + y^2 = 4}$

$r = 4\sin\theta, \quad 0 \le \theta \le \pi$



$r = 4\cos\theta : -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

(4,0)

Region we want:

$0 \le \theta \le \pi/4$

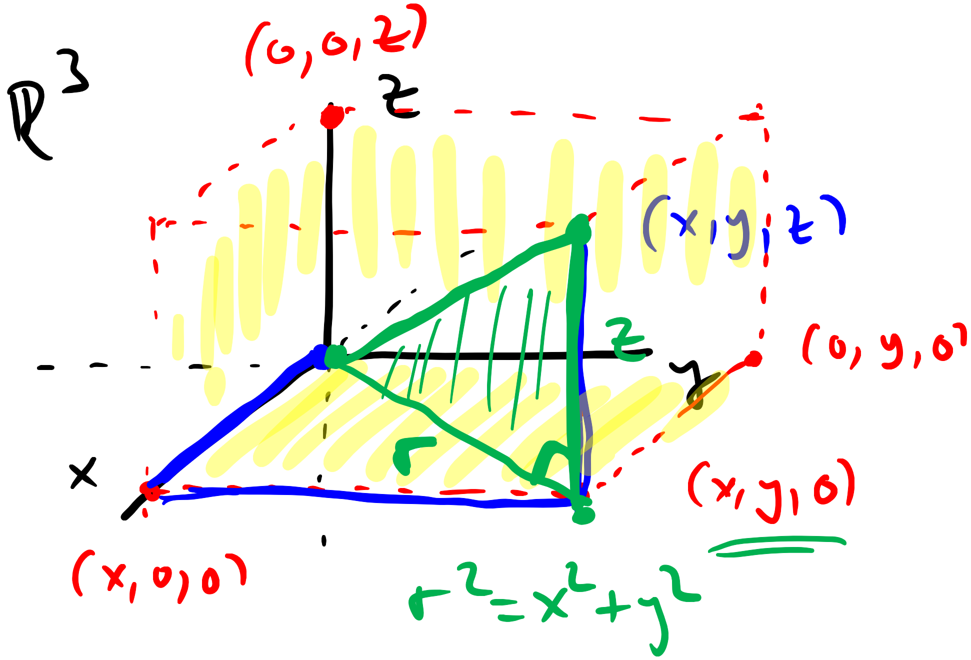
$4\sin\theta \le r \le 4\cos\theta$

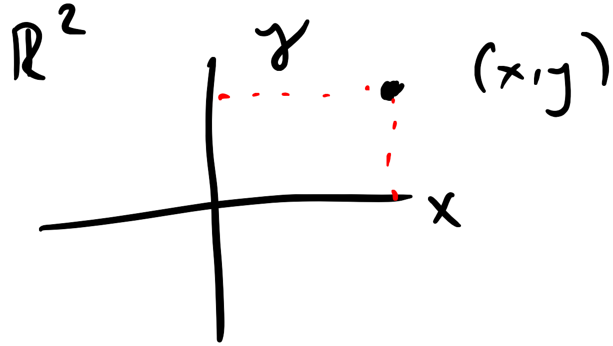
$$A = \underbrace{\frac{1}{2}\int_0^{\pi/4} (4\cos\theta)^2 \theta}_{\text{inside } r = 4\cos\theta} - \frac{1}{2}\int_0^{\frac{\pi}{4}} (4\sin\theta)^2 d\theta$$

$$= 8\int_0^{\pi/4} \underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos 2\theta} d\theta$$

$$= 4\sin 2\theta \Big|_0^{\pi/4} = 4$$

Cartesian coordinates in 3D





Coordinate planes

xy : $z = 0$

yz : $x = 0$

xz : $y = 0$

Distance: between (x_1, y_1, z_1) and (x_2, y_2, z_2)

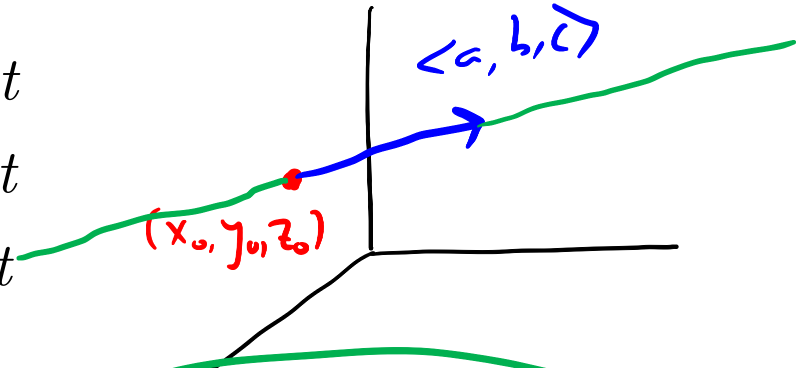
$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Lines and planes

Lines in $\mathbb{R}^3$ are definited using parametric equations:

$$\underbrace{\langle x, y, z\rangle = \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle}_{\vec{r}(t)}$$

$$x = x_0 + at$$
$$y = y_0 + bt$$
$$z = z_0 + ct$$

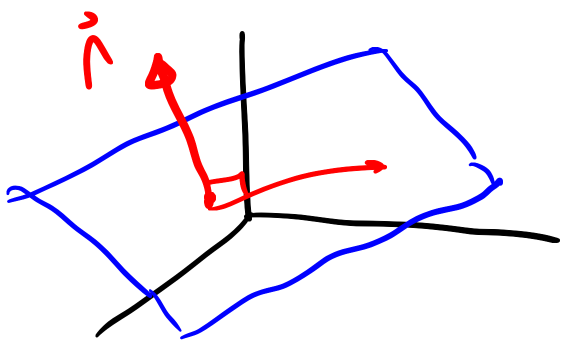


Planes are defined by a single linear relation: $ax + by + cz = d$. If you've done Math 1410, you've seen how to describe these using vectors.

$ax + by + cz = d$ (level surface: $g(x,y,z) = d$)

$\vec{n} = \langle a, b, c\rangle$

or $z = z_0 + Ax + By$ (graph: $z = f(x,y)$)

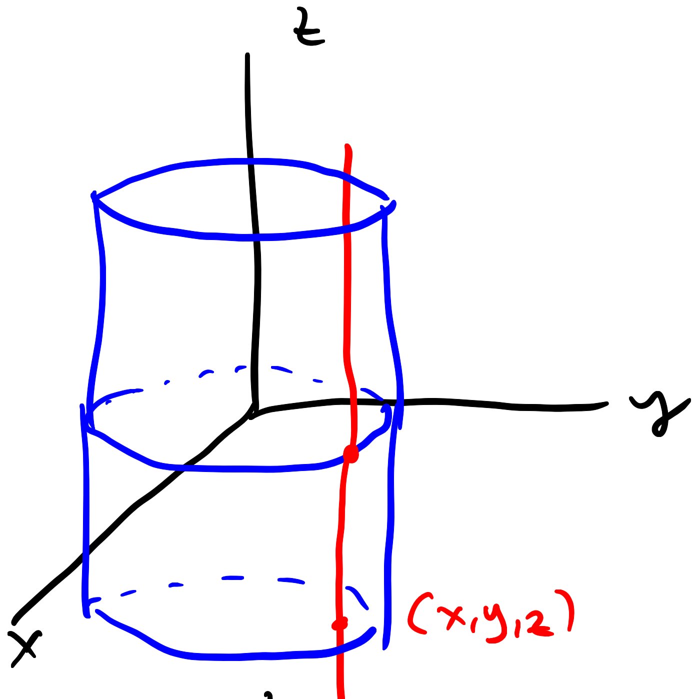


Cylinders

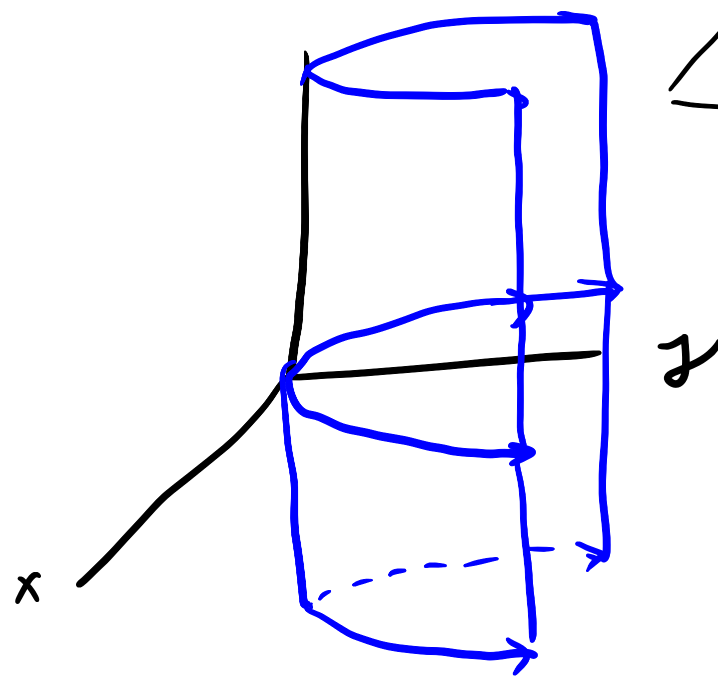
plotting resource: CalcPlot3D

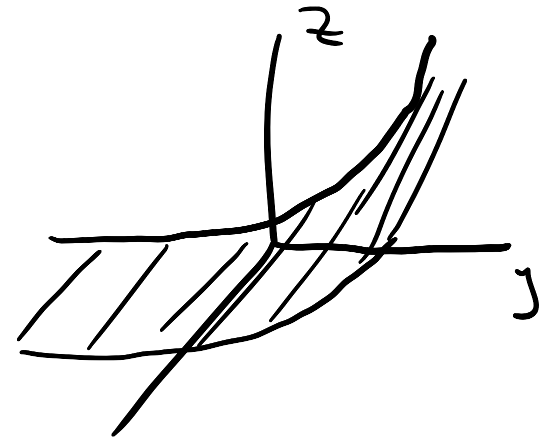
A *cylinder* is a surface where the *traces* (cross-sections) parallel to a given plane are all the same curve. Typically these come from an equation involving only two of three variables. For example:

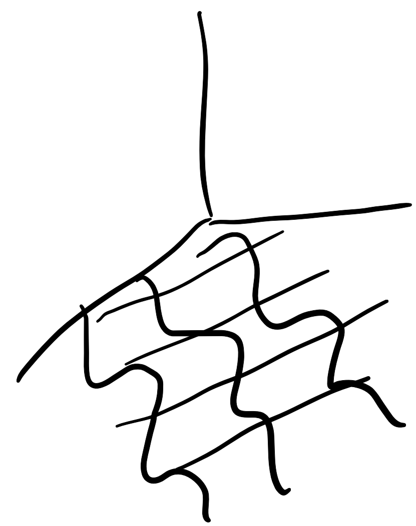
$$y = x^2 \quad z = e^y \quad z = \sin(x + y)$$



$x^2+y^2=1$







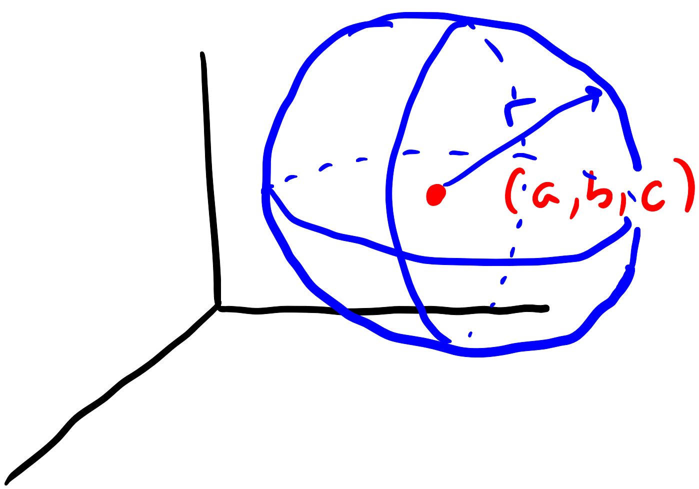
Quadric surfaces

These are the 3D analogues of the conic sections. Each one is defined by a quadratic equation in three variables.

Example

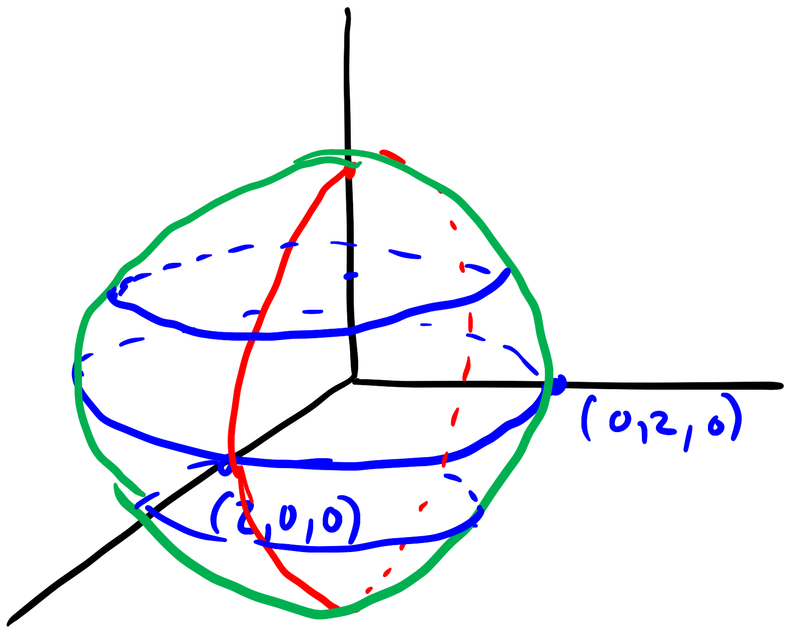
Describe the surface

$$\underbrace{(x-a)^2+(y-b)^2+(z-c)^2}_{(\text{distance})^2}=r^2.$$



Traces

Traces are curves that lie on surfaces. These curves are formed by intersecting a surface with a plane. Usually we consider traces with respect to planes parallel to the coordinate planes. These are the easiest to find, and are key to helping us visualize a surface.



$$x^2+y^2+z^2=4$$

$z=0$: $x^2+y^2=4$

$y=0$: $x^2+z^2=4$

$x=0$: $y^2+z^2=4$

$z=\pm 1$ $\quad x^2+y^2+1=4$

$x^2+y^2=3$

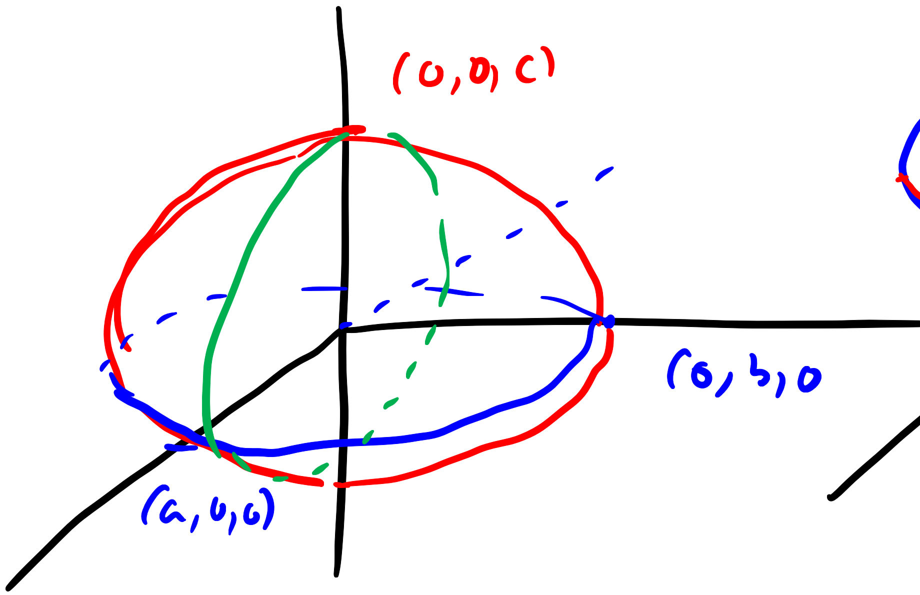
Ellipsoids (centre at (0,0,0))

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

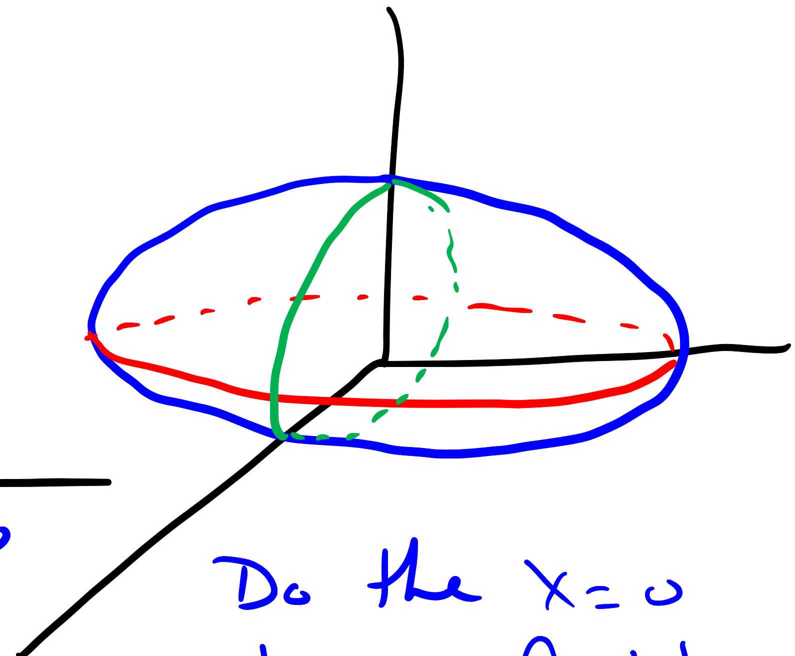
$z=0$

$x=0$

$y=0$



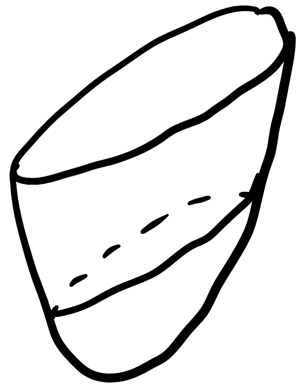
Bleah.



Do the $x=0$ trace first!

Elliptic paraboloids

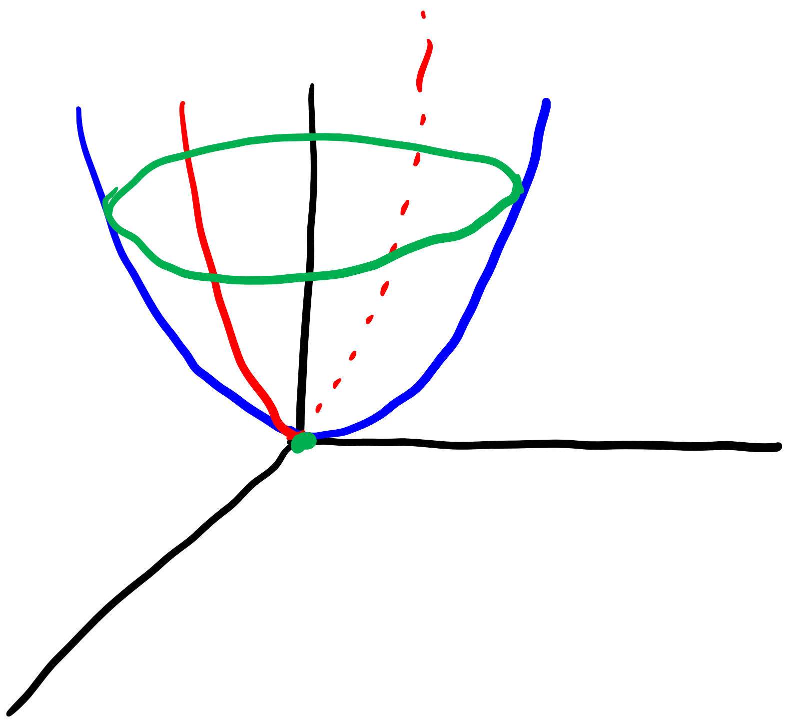
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$



$x = 0$

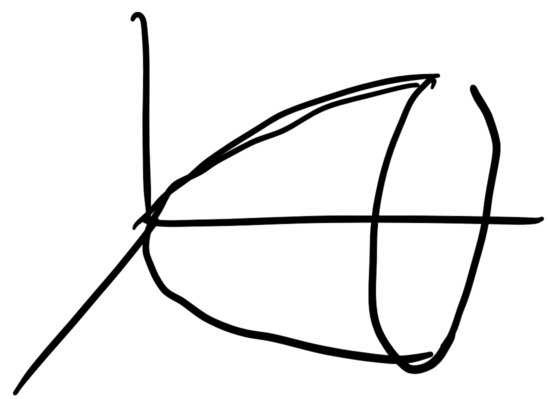
$y = 0$

$z = k$



also:

$$y = \frac{x^2}{a^2} + \frac{z^2}{b^2}$$



Hyperbolic paraboloids

Maintaining with tradition, we will now watch Sean try (and fail) to sketch a hyperbolic paraboloid.

$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

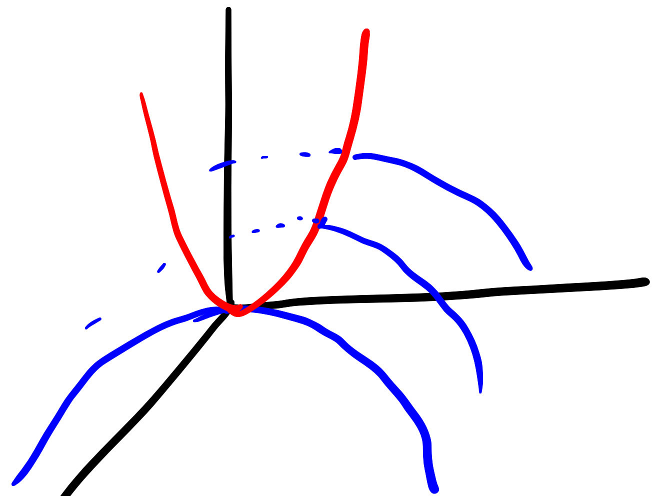
$(c > 0)$

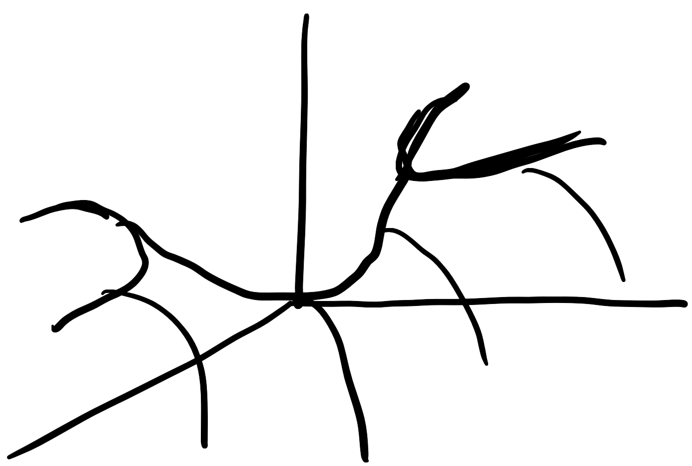
$z = k$

(hyperbola)

$x = 0$

$y = 0$

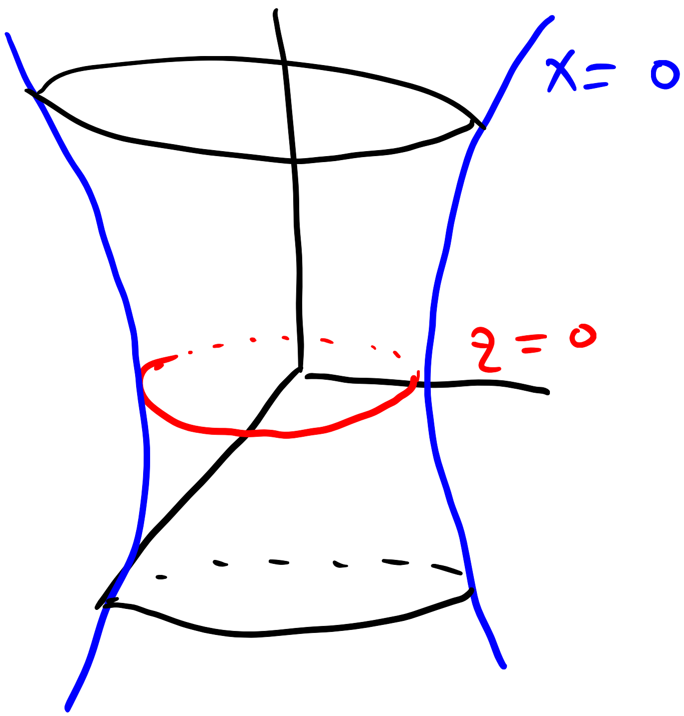


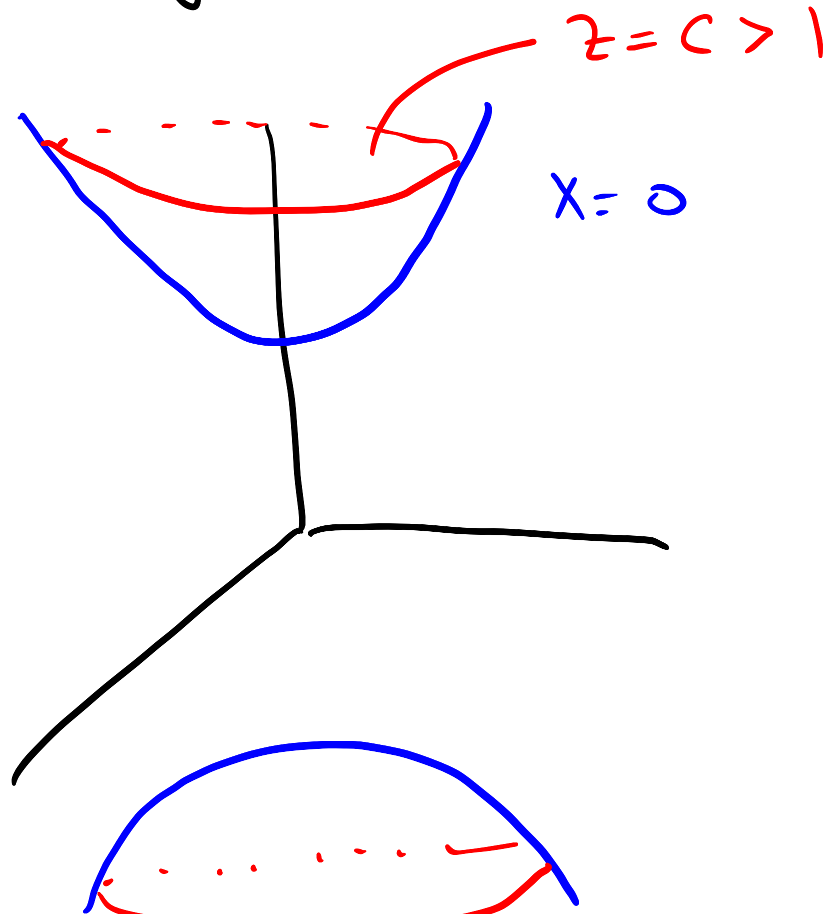


Hyperboloids

$x^2+y^2-z^2=1$

$-x^2-y^2+z^2=1$





Cones

Surfaces in general

More general surfaces in $\mathbb{R}^3$ can be obtained in several ways. Three common ones:

As a *graph* $z = f(x, y)$

As a *level surface* $f(x, y, z) = k$

As a *parametric surface* $x = f(u, v), y = g(u, v), z = h(u, v)$.

In Math 2565 we look mostly at the first case, and not at all at the third.

Functions of two variables

A function f of two variables takes a point $(x, y) \in \mathbb{R}^2$ as an input, and gives a point $z = f(x, y)$ as an output. If $D \subseteq \mathbb{R}^2$ is the domain of f we might write $f : D \to \mathbb{R}$ to emphasise the types of input and output.
Examples:

Examples

For the following functions, compute the values at $(0, 0), (1, 2)$, and $(3, -2)$. Determine the domain and range.

$$f(x, y) = x^2 - y^2$$

$$f(x, y) = e^{-(x^2+y^2)}$$

$$f(x, y) = \ln(xy)$$

Graphing functions of two variables

How do we even begin to visualize a function of two variables? (Or three, or four, or...) The key is to consider traces of the graph $z = f(x, y)$. Traces in planes $z = k$ are especially useful. These are called *level curves*. Examples:

$$z = x - y^2 \quad z = 4x^2 + y^2 \quad z = \sin(x)\cos(y)$$

Level curves vs. contour plots

You'll see mentions of both level curves and contour plots. These are related, but different. A level curve lies *on the surface*, in a plane $z = k$. A contour curve is the curve we get in the xy plane if we set $z = 0$. A contour plot is a collection of contour curves $f(x, y) = k$ for different values of k.