Math 2560 Calculus II

Fall 2018 Edition, University of Lethbridge

An adaptation of the APEX Calculus textbook, edited by Sean Fitzpatrick

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Except Chapter 4 (Differential Equations)

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This version of the text was assembled and edited by Sean Fitzpatrick, University of Lethbridge, November 2015, revised most recently in May 2018.
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Preface

This a custom textbook that covers the entire curriculum (as of September 2017) for the course Math 2560 (Calculus II) at the University of Lethbridge at minimal cost to the student. It is also an *Open Education Resource*. As a student, you are free to keep as many copies as you want, for as long as you want. You can print it, in whole or in part, or share it with a friend. As an instructor, I am free to modify the content as I see fit, whether this means editing to fit our curriculum, or simply correcting typos.

Most of this textbook is adapted from the *APEX Calculus* textbook project, which originated in the Department of Applied Mathematics at the Virginia Military Institute. (See apexcalculus.com.) On the following page you'll find the original preface from their text, which explains their project in more detail. They have produced calculus textbook that is **free** in two regards: it's free to download from their website, and the authors have made all the files needed to produce the textbook freely available, allowing others (such as myself) to edit the text to suit the needs of various courses (such as Math 2560).

What's even better is that the textbook is of remarkably high production quality: unlike many free texts, it is polished and professionally produced, with graphics on almost every page, and a large collection of exercises (with selected answers!).

I hope that you find this textbook useful. If you find any errors, or if you have any suggestions as to how the material could be better arranged or presented, please let me know. (The great thing about an open source textbook is that it can be edited at any time!) In particular, if you find a particular topic that you think needs further explanation, or more examples, or more exercises, please let us know. My hope is that this text will be improved every time it is used for this course.

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May, 2018

Preface to APEX Calculus

A Note on Using this Text

Thank you for reading this short preface. Allow us to share a few key points about the text so that you may better understand what you will find beyond this page.

This text is Part I of a three–text series on Calculus. The first part covers material taught in many "Calc 1" courses: limits, derivatives, and the basics of integration, found in Chapters 1 through 6.1. The second text covers material often taught in "Calc 2:" integration and its applications, along with an introduction to sequences, series and Taylor Polynomials, found in Chapters 5 through 8. The third text covers topics common in "Calc 3" or "multivariable calc:" parametric equations, polar coordinates, vector–valued functions, and functions of more than one variable, found in Chapters 9 through 13. All three are available separately for free at `www.apexcalculus.com`. These three texts are intended to work together and make one cohesive text, *APEX Calculus*, which can also be downloaded from the website.

Printing the entire text as one volume makes for a large, heavy, cumbersome book. One can certainly only print the pages they currently need, but some prefer to have a nice, bound copy of the text. Therefore this text has been split into these three manageable parts, each of which can be purchased for under $15 at Amazon.com.

A result of this splitting is that sometimes a concept is said to be explored in a "later section," though that section does not actually appear in this particular text. Also, the index makes reference to topics and page numbers that do not appear in this text. This is done intentionally to show the reader what topics are available for study. Downloading the .pdf of *APEX Calculus* will ensure that you have all the content.

For Students: How to Read this Text

Mathematics textbooks have a reputation for being hard to read. High–level mathematical writing often seeks to say much with few words, and this style often seeps into texts of lower–level topics. This book was written with the goal of being easier to read than many other calculus textbooks, without becoming too verbose.

Each chapter and section starts with an introduction of the coming material, hopefully setting the stage for "why you should care," and ends with a look ahead to see how the just–learned material helps address future problems.

Please read the text; it is written to explain the concepts of Calculus. There are numerous examples to demonstrate the meaning of definitions, the truth of theorems, and the application of mathematical techniques. When you encounter a sentence you don't understand, read it again. If it still doesn't make sense, read on anyway, as sometimes confusing sentences are explained by later sentences.

You don't have to read every equation. The examples generally show "all" the steps needed to solve a problem. Sometimes reading through each step is helpful; sometimes it is confusing. When the steps are illustrating a new technique, one probably should follow each step closely to learn the new technique. When the steps are showing the mathematics needed to find a number to be used later, one can usually skip ahead and see how that number is being used, instead of getting bogged down in reading how the number was found.

Most proofs have been omitted. In mathematics, *proving* something is always true is extremely important, and entails much more than testing to see if it works twice. However, students often are confused by the details of a proof, or become concerned that they should have been able to construct this proof on their own. To alleviate this potential problem, we do not include the proofs to most theorems in the text. The interested reader is highly encouraged to find proofs online or from their instructor. In most cases, one is very capable of understanding what a theorem *means* and *how to apply it* without knowing fully *why* it is true.

Interactive, 3D Graphics

New to Version 3.0 is the addition of interactive, 3D graphics in the .pdf version. Nearly all graphs of objects in space can be rotated, shifted, and zoomed in/out so the reader can better understand the object illustrated.

As of this writing, the only pdf viewers that support these 3D graphics are Adobe Reader & Acrobat (and only the versions for PC/Mac/Unix/Linux computers, not tablets or smartphones). To activate the interactive mode, click on the image. Once activated, one can click/drag to rotate the object and use the scroll wheel on a mouse to zoom in/out. (A great way to investigate an image is to first zoom in on the page of the pdf viewer so the graphic itself takes up much of the screen, then zoom inside the graphic itself.) A CTRL-click/drag pans the object left/right or up/down. By right-clicking on the graph one can access a menu of other options, such as changing the lighting scheme or perspective. One can also revert the graph back to its default view. If you wish to deactivate the interactivity, one can right-click and choose the “Disable Content” option.

Thanks

There are many people who deserve recognition for the important role they have played in the development of this text. First, I thank Michelle for her support and encouragement, even as this “project from work” occupied my time and attention at home. Many thanks to Troy Siemers, whose most important contributions extend far beyond the sections he wrote or the 227 figures he coded in Asymptote for 3D interaction. He provided incredible support, advice and encouragement for which I am very grateful. My thanks to Brian Heinold and Dimplekumar Chalishajar for their contributions and to Jennifer Bowen for reading through so much material and providing great feedback early on. Thanks to Troy, Lee Dewald, Dan Joseph, Meagan Herald, Bill Lowe, John David, Vonda Walsh, Geoff Cox, Jessica Libertini and other faculty of VMI who have given me numerous suggestions and corrections based on their experience with teaching from the text. (Special thanks to Troy, Lee & Dan for their patience in teaching Calc III while I was still writing the Calc III material.) Thanks to Randy Cone for encouraging his tutors of VMI’s Open Math Lab to read through the text and check the solutions, and thanks to the tutors for spending their time doing so. A very special thanks to Kristi Brown and Paul Janiczek who took this opportunity far above & beyond what I expected, meticulously checking every solution and carefully reading every example. Their comments have been extraordinarily helpful. I am also thankful for the support provided by Wane Schneiter, who as my Dean provided me with extra time to work on this project. I am blessed to have so many people give of their time to make this book better.

APEX – Affordable Print and Electronic teXts

APEX is a consortium of authors who collaborate to produce high–quality, low–cost textbooks. The current textbook–writing paradigm is facing a potential revolution as desktop publishing and electronic formats increase in popularity. However, writing a good textbook is no easy task, as the time requirements alone are substantial. It takes countless hours of work to produce text, write examples and exercises, edit and publish. Through collaboration, however, the cost to any individual can be lessened, allowing us to create texts that we freely distribute electronically and sell in printed form for an incredibly low cost. Having said that, nothing is entirely free; someone always bears some cost. This text "cost" the authors of this book their time, and that was not enough. *APEX Calculus* would not exist had not the Virginia Military Institute, through a generous Jackson–Hope grant, given the lead author significant time away from teaching so he could focus on this text.

Each text is available as a free .pdf, protected by a Creative Commons Attribution - Noncommercial 4.0 copyright. That means you can give the .pdf to anyone you like, print it in any form you like, and even edit the original content and redistribute it. If you do the latter, you must clearly reference this work and you cannot sell your edited work for money.

We encourage others to adapt this work to fit their own needs. One might add sections that are "missing" or remove sections that your students won't need. The source files can be found at `github.com/APEXCalculus`.

You can learn more at `www.vmi.edu/APEX`.

Version 4.0

Key changes from Version 3.0 to 4.0:

- Numerous typographical and "small" mathematical corrections (again, thanks to all my close readers!).
- "Large" mathematical corrections and adjustments. There were a number of places in Version 3.0 where a definition/theorem was not correct as stated. See `www.apexcalculus.com` for more information.
- More useful numbering of Examples, Theorems, etc. "Definition 11.4.2" refers to the second definition of Chapter 11, Section 4.
- The addition of Section 13.7: Triple Integration with Cylindrical and Spherical Coordinates
- The addition of Chapter 14: Vector Analysis.

6: Techniques of Antidifferentiation

In Calculus I you learned techniques that allow you to compute the derivative of practically any function you can conceive of creating using the elementary functions (polynomial, rational, exponential, logarithmic, trigonometric, etc.). You also learned how to define integration using Riemann sums, and saw how the Fundamental Theorem of Calculus relates integration to the antiderivative.

Computing antiderivatives is generally more difficult than computing derivatives. As an example, finding the derivative of $f(x) = x^2 \sin x$ is simple but we do not yet know how to find an antiderivative of f. Worse, we can find the derivative of $y = e^{x^2}$, but its antiderivatives *cannot* be written in terms of elementary functions.

Despite this latter difficulty, there are still broad classes of functions for which we can find antiderivatives. This chapter is dedicated to learning techniques to enable us to compute the antiderivatives of a wide variety of functions.

6.1 Substitution

We motivate this section with an example. Let $f(x) = (x^2 + 3x - 5)^{10}$. We can compute $f'(x)$ using the Chain Rule. It is:

$$f'(x) = 10(x^2 + 3x - 5)^9 \cdot (2x + 3) = (20x + 30)(x^2 + 3x - 5)^9.$$

Now consider this: What is $\int (20x + 30)(x^2 + 3x - 5)^9 \, dx$? We have the answer in front of us;

$$\int (20x + 30)(x^2 + 3x - 5)^9 \, dx = (x^2 + 3x - 5)^{10} + C.$$

How would we have evaluated this indefinite integral without starting with $f(x)$ as we did?

This section explores *integration by substitution.* It allows us to "undo the Chain Rule." Substitution allows us to evaluate the above integral without knowing the original function first.

The underlying principle is to rewrite a "complicated" integral of the form $\int f(x) \, dx$ as a not–so–complicated integral $\int h(u) \, du$. We'll formally establish later how this is done. First, consider again our introductory indefinite integral, $\int (20x + 30)(x^2 + 3x - 5)^9 \, dx$. Arguably the most "complicated" part of the integrand is $(x^2 + 3x - 5)^9$. We wish to make this simpler; we do so through a substitution. Let $u = x^2 + 3x - 5$. Thus

$$(x^2 + 3x - 5)^9 = u^9.$$

We have established u as a function of x, so now consider the differential of u:

$$du = (2x + 3)dx.$$

Keep in mind that $(2x+3)$ and dx are multiplied; the dx is not "just sitting there."

Return to the original integral and do some substitutions through algebra:

$$\begin{aligned}\int (20x+30)(x^2+3x-5)^9\,dx &= \int 10(2x+3)(x^2+3x-5)^9\,dx\\ &= \int 10(\underbrace{x^2+3x-5}_{u})^9\,\underbrace{(2x+3)\,dx}_{du}\\ &= \int 10u^9\,du\\ &= u^{10}+C \quad \text{(replace } u \text{ with } x^2+3x-5)\\ &= (x^2+3x-5)^{10}+C\end{aligned}$$

One might well look at this and think "I (sort of) followed how that worked, but I could never come up with that on my own," but the process can be learned. This section contains numerous examples through which the reader will gain understanding and mathematical maturity enabling them to regard substitution as a natural tool when evaluating integrals.

We stated before that integration by substitution "undoes" the Chain Rule. Specifically, let $F(x)$ and $g(x)$ be differentiable functions and consider the derivative of their composition:

$$\frac{d}{dx}\Big(F\big(g(x)\big)\Big) = F'(g(x))g'(x).$$

Thus

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

Integration by substitution works by recognizing the "inside" function $g(x)$ and replacing it with a variable. By setting $u = g(x)$, we can rewrite the derivative as

$$\frac{d}{dx}\Big(F(u)\Big) = F'(u)u'.$$

Since $du = g'(x)dx$, we can rewrite the above integral as

$$\int F'(g(x))g'(x)\,dx = \int F'(u)du = F(u) + C = F(g(x)) + C.$$

This concept is important so we restate it in the context of a theorem.

Theorem 6.1.1 Integration by Substitution

Let F and g be differentiable functions, where the range of g is an interval I contained in the domain of F. Then

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C.$$

If $u = g(x)$, then $du = g'(x)dx$ and

$$\int F'(g(x))g'(x)\,dx = \int F'(u)\,du = F(u) + C = F(g(x)) + C.$$

The point of substitution is to make the integration step easy. Indeed, the step $\int F'(u)\,du = F(u)+C$ looks easy, as the antiderivative of the derivative of F is just F, plus a constant. The "work" involved is making the proper substitution.

There is not a step–by–step process that one can memorize; rather, experience will be one's guide. To gain experience, we now embark on many examples.

Example 6.1.1 Integrating by substitution

Evaluate $\displaystyle\int x\sin(x^2+5)\,dx$.

Solution Knowing that substitution is related to the Chain Rule, we choose to let u be the "inside" function of $\sin(x^2+5)$. (This is not *always* a good choice, but it is often the best place to start.)

Let $u = x^2+5$, hence $du = 2x\,dx$. The integrand has an $x\,dx$ term, but not a $2x\,dx$ term. (Recall that multiplication is commutative, so the x does not physically have to be next to dx for there to be an $x\,dx$ term.) We can divide both sides of the du expression by 2:

$$du = 2x\,dx \quad \Rightarrow \quad \frac{1}{2}du = x\,dx.$$

We can now substitute.

$$\begin{aligned}\int x\sin(x^2+5)\,dx &= \int \sin(\underbrace{x^2+5}_{u})\,\underbrace{x\,dx}_{\frac{1}{2}du}\\ &= \int \frac{1}{2}\sin u\,du\\ &= -\frac{1}{2}\cos u + C \quad \text{(now replace } u \text{ with } x^2+5\text{)}\\ &= -\frac{1}{2}\cos(x^2+5)+C.\end{aligned}$$

Thus $\int x\sin(x^2+5)\,dx = -\frac{1}{2}\cos(x^2+5)+C$. We can check our work by evaluating the derivative of the right hand side.

Example 6.1.2 Integrating by substitution

Evaluate $\displaystyle\int \cos(5x)\,dx$.

Solution Again let u replace the "inside" function. Letting $u = 5x$, we have $du = 5dx$. Since our integrand does not have a $5dx$ term, we can divide the previous equation by 5 to obtain $\frac{1}{5}du = dx$. We can now substitute.

$$\begin{aligned}\int \cos(5x)\,dx &= \int \cos(\underbrace{5x}_{u})\,\underbrace{dx}_{\frac{1}{5}du}\\ &= \int \frac{1}{5}\cos u\,du\\ &= \frac{1}{5}\sin u + C\\ &= \frac{1}{5}\sin(5x)+C.\end{aligned}$$

We can again check our work through differentiation.

The previous example exhibited a common, and simple, type of substitution. The "inside" function was a linear function (in this case, $y = 5x$). When the inside function is linear, the resulting integration is very predictable, outlined here.

Key Idea 6.1.1 Substitution With A Linear Function

Consider $\int F'(ax+b)\,dx$, where $a \neq 0$ and b are constants. Letting $u = ax+b$ gives $du = a\cdot dx$, leading to the result

$$\int F'(ax+b)\,dx = \frac{1}{a}F(ax+b)+C.$$

Thus $\int \sin(7x-4)\,dx = -\frac{1}{7}\cos(7x-4)+C$. Our next example can use Key Idea 6.1.1, but we will only employ it after going through all of the steps.

Example 6.1.3 Integrating by substituting a linear function

Evaluate $\displaystyle\int \frac{7}{-3x+1}\,dx$.

Solution View the integrand as the composition of functions $f(g(x))$, where $f(x) = 7/x$ and $g(x) = -3x+1$. Employing our understanding of substitution, we let $u = -3x+1$, the inside function. Thus $du = -3dx$. The integrand lacks a -3; hence divide the previous equation by -3 to obtain $-du/3 = dx$. We can now evaluate the integral through substitution.

$$\begin{aligned}\int \frac{7}{-3x+1}\,dx &= \int \frac{7}{u}\frac{du}{-3}\\ &= \frac{-7}{3}\int\frac{du}{u}\\ &= \frac{-7}{3}\ln|u|+C\\ &= -\frac{7}{3}\ln|-3x+1|+C.\end{aligned}$$

Using Key Idea 6.1.1 is faster, recognizing that u is linear and $a = -3$. One may want to continue writing out all the steps until they are comfortable with this particular shortcut.

Not all integrals that benefit from substitution have a clear "inside" function. Several of the following examples will demonstrate ways in which this occurs.

Example 6.1.4 Integrating by substitution

Evaluate $\displaystyle\int \sin x\cos x\,dx$.

Solution There is not a composition of function here to exploit; rather, just a product of functions. Do not be afraid to experiment; when given an integral to evaluate, it is often beneficial to think "If I let u be *this*, then du must be *that* …" and see if this helps simplify the integral at all.

In this example, let's set $u = \sin x$. Then $du = \cos x\,dx$, which we have as part of the integrand! The substitution becomes very straightforward:

$$\begin{aligned}\int \sin x \cos x\, dx &= \int u\, du\\ &= \frac{1}{2}u^2 + C\\ &= \frac{1}{2}\sin^2 x + C.\end{aligned}$$

One would do well to ask "What would happen if we let $u = \cos x$?" The result is just as easy to find, yet looks very different. The challenge to the reader is to evaluate the integral letting $u = \cos x$ and discover why the answer is the same, yet looks different.

Our examples so far have required "basic substitution." The next example demonstrates how substitutions can be made that often strike the new learner as being "nonstandard."

Example 6.1.5 Integrating by substitution

Evaluate $\displaystyle\int x\sqrt{x+3}\, dx$.

Solution Recognizing the composition of functions, set $u = x + 3$. Then $du = dx$, giving what seems initially to be a simple substitution. But at this stage, we have:

$$\int x\sqrt{x+3}\, dx = \int x\sqrt{u}\, du.$$

We cannot evaluate an integral that has both an x and an u in it. We need to convert the x to an expression involving just u.

Since we set $u = x+3$, we can also state that $u-3 = x$. Thus we can replace x in the integrand with $u - 3$. It will also be helpful to rewrite $\sqrt{u}$ as $u^{\frac{1}{2}}$.

$$\begin{aligned}\int x\sqrt{x+3}\, dx &= \int (u-3)u^{\frac{1}{2}}\, du\\ &= \int \left(u^{\frac{3}{2}} - 3u^{\frac{1}{2}}\right) du\\ &= \frac{2}{5}u^{\frac{5}{2}} - 2u^{\frac{3}{2}} + C\\ &= \frac{2}{5}(x+3)^{\frac{5}{2}} - 2(x+3)^{\frac{3}{2}} + C.\end{aligned}$$

Checking your work is always a good idea. In this particular case, some algebra will be needed to make one's answer match the integrand in the original problem.

Example 6.1.6 Integrating by substitution

Evaluate $\displaystyle\int \frac{1}{x\ln x}\, dx$.

Solution This is another example where there does not seem to be an obvious composition of functions. The line of thinking used in Example 6.1.5 is useful here: choose something for u and consider what this implies du must be. If u can be chosen such that du also appears in the integrand, then we have chosen well.

Choosing $u = 1/x$ makes $du = -1/x^2\,dx$; that does not seem helpful. However, setting $u = \ln x$ makes $du = 1/x\,dx$, which is part of the integrand. Thus:

$$\begin{aligned}\int \frac{1}{x\ln x}\,dx &= \int \underbrace{\frac{1}{\ln x}}_{u}\underbrace{\frac{1}{x}\,dx}_{du}\\ &= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln|\ln x| + C.\end{aligned}$$

The final answer is interesting; the natural log of the natural log. Take the derivative to confirm this answer is indeed correct.

Integrals Involving Trigonometric Functions

Section 6.3 delves deeper into integrals of a variety of trigonometric functions; here we use substitution to establish a foundation that we will build upon.

The next three examples will help fill in some missing pieces of our antiderivative knowledge. We know the antiderivatives of the sine and cosine functions; what about the other standard functions tangent, cotangent, secant and cosecant? We discover these next.

Example 6.1.7 Integration by substitution: antiderivatives of $\tan x$

Evaluate $\displaystyle\int \tan x\,dx$.

Solution The previous paragraph established that we did not know the antiderivatives of tangent, hence we must assume that we have learned something in this section that can help us evaluate this indefinite integral.

Rewrite $\tan x$ as $\sin x/\cos x$. While the presence of a composition of functions may not be immediately obvious, recognize that $\cos x$ is "inside" the $1/x$ function. Therefore, we see if setting $u = \cos x$ returns usable results. We have that $du = -\sin x\,dx$, hence $-du = \sin x\,dx$. We can integrate:

$$\begin{aligned}\int \tan x\,dx &= \int \frac{\sin x}{\cos x}\,dx\\ &= \int \underbrace{\frac{1}{\cos x}}_{u}\underbrace{\sin x\,dx}_{-du}\\ &= \int \frac{-1}{u}\,du\\ &= -\ln|u| + C\\ &= -\ln|\cos x| + C.\end{aligned}$$

Some texts prefer to bring the -1 inside the logarithm as a power of $\cos x$, as in:

$$\begin{aligned}-\ln|\cos x| + C &= \ln|(\cos x)^{-1}| + C\\ &= \ln\left|\frac{1}{\cos x}\right| + C\\ &= \ln|\sec x| + C.\end{aligned}$$

Thus the result they give is $\int \tan x\,dx = \ln|\sec x| + C$. These two answers are equivalent.

Example 6.1.8 Integrating by substitution: antiderivatives of $\sec x$

Evaluate $\displaystyle\int \sec x\, dx$.

Solution This example employs a wonderful trick: multiply the integrand by "1" so that we see how to integrate more clearly. In this case, we write "1" as

$$1 = \frac{\sec x + \tan x}{\sec x + \tan x}.$$

This may seem like it came out of left field, but it works beautifully. Consider:

$$\begin{aligned}\int \sec x\, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx\\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx.\end{aligned}$$

Now let $u = \sec x + \tan x$; this means $du = (\sec x \tan x + \sec^2 x)\, dx$, which is our numerator. Thus:

$$\begin{aligned}&= \int \frac{du}{u}\\ &= \ln|u| + C\\ &= \ln|\sec x + \tan x| + C.\end{aligned}$$

We can use similar techniques to those used in Examples 6.1.7 and 6.1.8 to find antiderivatives of $\cot x$ and $\csc x$ (which the reader can explore in the exercises.) We summarize our results here.

Theorem 6.1.2 Antiderivatives of Trigonometric Functions

1. $\displaystyle\int \sin x\, dx = -\cos x + C$
2. $\displaystyle\int \cos x\, dx = \sin x + C$
3. $\displaystyle\int \tan x\, dx = -\ln|\cos x| + C$
4. $\displaystyle\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
5. $\displaystyle\int \sec x\, dx = \ln|\sec x + \tan x| + C$
6. $\displaystyle\int \cot x\, dx = \ln|\sin x| + C$

We explore one more common trigonometric integral.

Example 6.1.9 Integration by substitution: powers of $\cos x$ and $\sin x$

Evaluate $\displaystyle\int \cos^2 x\, dx$.

Solution We have a composition of functions as $\cos^2 x = \big(\cos x\big)^2$. However, setting $u = \cos x$ means $du = -\sin x\, dx$, which we do not have in the integral. Another technique is needed.

The process we'll employ is to use a Power Reducing formula for $\cos^2 x$ (perhaps consult the back of this text for this formula), which states

$$\cos^2 x = \frac{1 + \cos(2x)}{2}.$$

The right hand side of this equation is not difficult to integrate. We have:

$$\int \cos^2 x\, dx = \int \frac{1+\cos(2x)}{2}\, dx$$
$$= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2x)\right)\, dx.$$

Now use Key Idea 6.1.1:

$$= \frac{1}{2}x + \frac{1}{2}\frac{\sin(2x)}{2} + C$$
$$= \frac{1}{2}x + \frac{\sin(2x)}{4} + C.$$

We'll make significant use of this power–reducing technique in future sections.

Simplifying the Integrand

It is common to be reluctant to manipulate the integrand of an integral; at first, our grasp of integration is tenuous and one may think that working with the integrand will improperly change the results. Integration by substitution works using a different logic: as long as *equality* is maintained, the integrand can be manipulated so that its *form* is easier to deal with. The next two examples demonstrate common ways in which using algebra first makes the integration easier to perform.

Example 6.1.10 Integration by substitution: simplifying first

Evaluate $\displaystyle\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\, dx.$

SOLUTION One may try to start by setting u equal to either the numerator or denominator; in each instance, the result is not workable.

When dealing with rational functions (i.e., quotients made up of polynomial functions), it is an almost universal rule that everything works better when the degree of the numerator is less than the degree of the denominator. Hence we use polynomial division.

We skip the specifics of the steps, but note that when x^2+2x+1 is divided into x^3+4x^2+8x+5, it goes in $x+2$ times with a remainder of $3x+3$. Thus

$$\frac{x^3+4x^2+8x+5}{x^2+2x+1} = x+2+\frac{3x+3}{x^2+2x+1}.$$

Integrating $x+2$ is simple. The fraction can be integrated by setting $u = x^2 + 2x+1$, giving $du = (2x+2)\, dx$. This is very similar to the numerator. Note that

$du/2 = (x+1)\,dx$ and then consider the following:

$$\begin{aligned}\int \frac{x^3+4x^2+8x+5}{x^2+2x+1}\,dx &= \int \left(x+2+\frac{3x+3}{x^2+2x+1}\right)\,dx \\ &= \int (x+2)\,dx + \int \frac{3(x+1)}{x^2+2x+1}\,dx \\ &= \frac{1}{2}x^2+2x+C_1+\int \frac{3}{u}\frac{du}{2} \\ &= \frac{1}{2}x^2+2x+C_1+\frac{3}{2}\ln|u|+C_2 \\ &= \frac{1}{2}x^2+2x+\frac{3}{2}\ln|x^2+2x+1|+C.\end{aligned}$$

In some ways, we "lucked out" in that after dividing, substitution was able to be done. In later sections we'll develop techniques for handling rational functions where substitution is not directly feasible.

Example 6.1.11 Integration by alternate methods

Evaluate $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx$ with, and without, substitution.

Solution We already know how to integrate this particular example. Rewrite $\sqrt{x}$ as $x^{\frac{1}{2}}$ and simplify the fraction:

$$\frac{x^2+2x+3}{x^{1/2}} = x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}.$$

We can now integrate using the Power Rule:

$$\begin{aligned}\int \frac{x^2+2x+3}{x^{1/2}}\,dx &= \int \left(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}}\right)\,dx \\ &= \frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}+C\end{aligned}$$

This is a perfectly fine approach. We demonstrate how this can also be solved using substitution as its implementation is rather clever.

Let $u=\sqrt{x}=x^{\frac{1}{2}}$; therefore

$$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2\sqrt{x}}\,dx \quad\Rightarrow\quad 2du = \frac{1}{\sqrt{x}}\,dx.$$

This gives us $\displaystyle\int \frac{x^2+2x+3}{\sqrt{x}}\,dx = \int (x^2+2x+3)\cdot 2\,du$. What are we to do with the other x terms? Since $u=x^{\frac{1}{2}}$, $u^2=x$, etc. We can then replace x^2 and x with appropriate powers of u. We thus have

$$\begin{aligned}\int \frac{x^2+2x+3}{\sqrt{x}}\,dx &= \int (x^2+2x+3)\cdot 2\,du \\ &= \int 2(u^4+2u^2+3)\,du \\ &= \frac{2}{5}u^5+\frac{4}{3}u^3+6u+C \\ &= \frac{2}{5}x^{\frac{5}{2}}+\frac{4}{3}x^{\frac{3}{2}}+6x^{\frac{1}{2}}+C,\end{aligned}$$

which is obviously the same answer we obtained before. In this situation, substitution is arguably more work than our other method. The fantastic thing is that it works. It demonstrates how flexible integration is.

Substitution and Inverse Trigonometric Functions

When studying derivatives of inverse functions, we learned that

$$\frac{d}{dx}\big(\tan^{-1} x\big) = \frac{1}{1+x^2}.$$

Applying the Chain Rule to this is not difficult; for instance,

$$\frac{d}{dx}\big(\tan^{-1} 5x\big) = \frac{5}{1+25x^2}.$$

We now explore how Substitution can be used to "undo" certain derivatives that are the result of the Chain Rule applied to Inverse Trigonometric functions. We begin with an example.

Example 6.1.12 Integrating by substitution: inverse trigonometric functions

Evaluate $\displaystyle\int \frac{1}{25+x^2}\,dx.$

Solution The integrand looks similar to the derivative of the arctangent function. Note:

$$\begin{aligned}\frac{1}{25+x^2} &= \frac{1}{25(1+\frac{x^2}{25})}\\ &= \frac{1}{25(1+\left(\frac{x}{5}\right)^2)}\\ &= \frac{1}{25}\frac{1}{1+\left(\frac{x}{5}\right)^2}.\end{aligned}$$

Thus

$$\int \frac{1}{25+x^2}\,dx = \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx.$$

This can be integrated using Substitution. Set $u = x/5$, hence $du = dx/5$ or $dx = 5du$. Thus

$$\begin{aligned}\int \frac{1}{25+x^2}\,dx &= \frac{1}{25}\int \frac{1}{1+\left(\frac{x}{5}\right)^2}\,dx\\ &= \frac{1}{5}\int \frac{1}{1+u^2}\,du\\ &= \frac{1}{5}\tan^{-1} u + C\\ &= \frac{1}{5}\tan^{-1}\left(\frac{x}{5}\right) + C\end{aligned}$$

Example 6.1.12 demonstrates a general technique that can be applied to other integrands that result in inverse trigonometric functions. The results are summarized here.

Theorem 6.1.3 Integrals Involving Inverse Trigonometric Functions

Let $a > 0$.

1. $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
2. $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
3. $\displaystyle\int \frac{1}{x\sqrt{x^2-a^2}}\,dx = \frac{1}{a}\sec^{-1}\left(\frac{|x|}{a}\right) + C$

Let's practice using Theorem 6.1.3.

Example 6.1.13 Integrating by substitution: inverse trigonometric functions

Evaluate the given indefinite integrals.

$$1.\ \int \frac{1}{9+x^2}\,dx \quad 2.\ \int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx \quad 3.\ \int \frac{1}{\sqrt{5-x^2}}\,dx.$$

Solution Each can be answered using a straightforward application of Theorem 6.1.3.

1. $\displaystyle\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$, as $a = 3$.
2. $\displaystyle\int \frac{1}{x\sqrt{x^2-\frac{1}{100}}}\,dx = 10\sec^{-1}10x + C$, as $a = \frac{1}{10}$.
3. $\displaystyle\int \frac{1}{\sqrt{5-x^2}} = \sin^{-1}\frac{x}{\sqrt{5}} + C$, as $a = \sqrt{5}$.

Most applications of Theorem 6.1.3 are not as straightforward. The next examples show some common integrals that can still be approached with this theorem.

Example 6.1.14 Integrating by substitution: completing the square

Evaluate $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$.

Solution Initially, this integral seems to have nothing in common with the integrals in Theorem 6.1.3. As it lacks a square root, it almost certainly is not related to arcsine or arcsecant. It is, however, related to the arctangent function.

We see this by *completing the square* in the denominator. We give a brief reminder of the process here.

Start with a quadratic with a leading coefficient of 1. It will have the form of x^2+bx+c. Take 1/2 of b, square it, and add/subtract it back into the expression. I.e.,

$$\begin{aligned} x^2+bx+c &= \underbrace{x^2+bx+\frac{b^2}{4}}_{(x+b/2)^2} - \frac{b^2}{4} + c \\ &= \left(x+\frac{b}{2}\right)^2 + c - \frac{b^2}{4} \end{aligned}$$

In our example, we take half of -4 and square it, getting 4. We add/subtract it into the denominator as follows:

$$\begin{aligned}\frac{1}{x^2-4x+13} &= \frac{1}{\underbrace{x^2-4x+4}_{(x-2)^2}-4+13}\\ &= \frac{1}{(x-2)^2+9}\end{aligned}$$

We can now integrate this using the arctangent rule. Technically, we need to substitute first with $u = x-2$, but we can employ Key Idea 6.1.1 instead. Thus we have

$$\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+9}\,dx = \frac{1}{3}\tan^{-1}\frac{x-2}{3} + C.$$

Example 6.1.15 Integrals requiring multiple methods

Evaluate $\displaystyle\int \frac{4-x}{\sqrt{16-x^2}}\,dx$.

Solution This integral requires two different methods to evaluate it. We get to those methods by splitting up the integral:

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = \int \frac{4}{\sqrt{16-x^2}}\,dx - \int \frac{x}{\sqrt{16-x^2}}\,dx.$$

The first integral is handled using a straightforward application of Theorem 6.1.3; the second integral is handled by substitution, with $u = 16-x^2$. We handle each separately.

$\displaystyle\int \frac{4}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + C.$

$\displaystyle\int \frac{x}{\sqrt{16-x^2}}\,dx$: Set $u = 16-x^2$, so $du = -2xdx$ and $xdx = -du/2$. We have

$$\begin{aligned}\int \frac{x}{\sqrt{16-x^2}}\,dx &= \int \frac{-du/2}{\sqrt{u}}\\ &= -\frac{1}{2}\int \frac{1}{\sqrt{u}}\,du\\ &= -\sqrt{u}+C\\ &= -\sqrt{16-x^2}+C.\end{aligned}$$

Combining these together, we have

$$\int \frac{4-x}{\sqrt{16-x^2}}\,dx = 4\sin^{-1}\frac{x}{4} + \sqrt{16-x^2} + C.$$

Substitution and Definite Integration

This section has focused on evaluating indefinite integrals as we are learning a new technique for finding antiderivatives. However, much of the time integration is used in the context of a definite integral. Definite integrals that require substitution can be calculated using the following workflow:

1. Start with a definite integral $\int_a^b f(x)\,dx$ that requires substitution.
2. Ignore the bounds; use substitution to evaluate $\int f(x)\,dx$ and find an antiderivative $F(x)$.
3. Evaluate $F(x)$ at the bounds; that is, evaluate $F(x)\Big|_a^b = F(b) - F(a)$.

This workflow works fine, but substitution offers an alternative that is powerful and amazing (and a little time saving).

At its heart, (using the notation of Theorem 6.1.1) substitution converts integrals of the form $\int F'(g(x))g'(x)\,dx$ into an integral of the form $\int F'(u)\,du$ with the substitution of $u = g(x)$. The following theorem states how the bounds of a definite integral can be changed as the substitution is performed.

Theorem 6.1.4 Substitution with Definite Integrals

Let F and g be differentiable functions, where the range of g is an interval I that is contained in the domain of F. Then

$$\int_a^b F'\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} F'(u)\,du.$$

In effect, Theorem 6.1.4 states that once you convert to integrating with respect to u, you do not need to switch back to evaluating with respect to x. A few examples will help one understand.

Example 6.1.16 Definite integrals and substitution: changing the bounds

Evaluate $\int_0^2 \cos(3x-1)\,dx$ using Theorem 6.1.4.

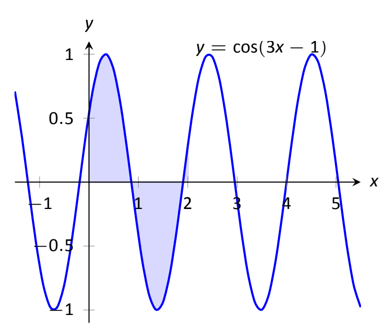
Solution Observing the composition of functions, let $u = 3x - 1$, hence $du = 3dx$. As $3dx$ does not appear in the integrand, divide the latter equation by 3 to get $du/3 = dx$.

By setting $u = 3x - 1$, we are implicitly stating that $g(x) = 3x - 1$. Theorem 6.1.4 states that the new lower bound is $g(0) = -1$; the new upper bound is $g(2) = 5$. We now evaluate the definite integral:

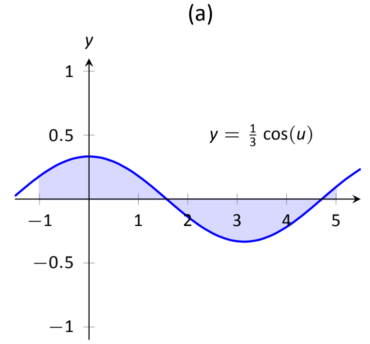
$$\begin{aligned}\int_0^2 \cos(3x-1)\,dx &= \int_{-1}^5 \cos u \frac{du}{3}\\ &= \frac{1}{3}\sin u\Big|_{-1}^5\\ &= \frac{1}{3}\big(\sin 5 - \sin(-1)\big) \approx -0.039.\end{aligned}$$

Notice how once we converted the integral to be in terms of u, we never went back to using x.

The graphs in Figure 6.1.1 tell more of the story. In (a) the area defined by the original integrand is shaded, whereas in (b) the area defined by the new integrand is shaded. In this particular situation, the areas look very similar; the new region is "shorter" but "wider," giving the same area.



(a)



(b)

Figure 6.1.1: Graphing the areas defined by the definite integrals of Example 6.1.16.

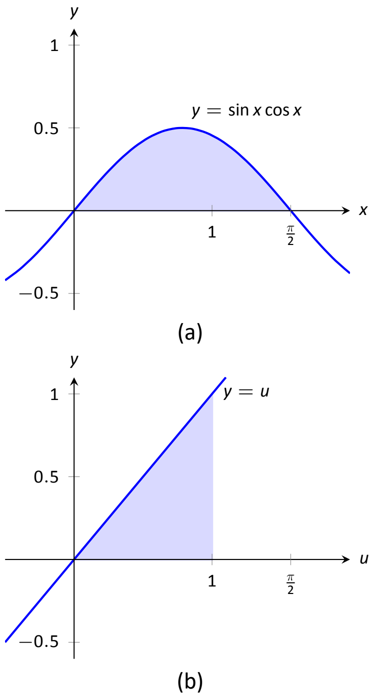


Figure 6.1.2: Graphing the areas defined by the definite integrals of Example 6.1.17.

Example 6.1.17 Definite integrals and substitution: changing the bounds

Evaluate $\displaystyle\int_0^{\pi/2} \sin x \cos x \, dx$ using Theorem 6.1.4.

Solution We saw the corresponding indefinite integral in Example 6.1.4. In that example we set $u = \sin x$ but stated that we could have let $u = \cos x$. For variety, we do the latter here.

Let $u = g(x) = \cos x$, giving $du = -\sin x \, dx$ and hence $\sin x \, dx = -du$. The new upper bound is $g(\pi/2) = 0$; the new lower bound is $g(0) = 1$. Note how the lower bound is actually larger than the upper bound now. We have

$$\begin{aligned}\int_0^{\pi/2} \sin x \cos x \, dx &= \int_1^0 -u \, du \quad \text{(switch bounds \& change sign)}\\ &= \int_0^1 u \, du\\ &= \frac{1}{2}u^2\Big|_0^1 = 1/2.\end{aligned}$$

In Figure 6.1.2 we have again graphed the two regions defined by our definite integrals. Unlike the previous example, they bear no resemblance to each other. However, Theorem 6.1.4 guarantees that they have the same area.

Integration by substitution is a powerful and useful integration technique. The next section introduces another technique, called Integration by Parts. As substitution "undoes" the Chain Rule, integration by parts "undoes" the Product Rule. Together, these two techniques provide a strong foundation on which most other integration techniques are based.

Exercises 6.1

Terms and Concepts

1. Substitution "undoes" what derivative rule?

2. T/F: One can use algebra to rewrite the integrand of an integral to make it easier to evaluate.

Problems

In Exercises 3 – 14, evaluate the indefinite integral to develop an understanding of Substitution.

3. $\int 3x^2\left(x^3-5\right)^7\,dx$

4. $\int (2x-5)\left(x^2-5x+7\right)^3\,dx$

5. $\int x\left(x^2+1\right)^8\,dx$

6. $\int (12x+14)\left(3x^2+7x-1\right)^5\,dx$

7. $\int \frac{1}{2x+7}dx$

8. $\int \frac{1}{\sqrt{2x+3}}dx$

9. $\int \frac{x}{\sqrt{x+3}}dx$

10. $\int \frac{x^3-x}{\sqrt{x}}dx$

11. $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}dx$

12. $\int \frac{x^4}{\sqrt{x^5+1}}dx$

13. $\int \frac{\frac{1}{x}+1}{x^2}dx$

14. $\int \frac{\ln(x)}{x}dx$

In Exercises 15 – 24, use Substitution to evaluate the indefinite integral involving trigonometric functions.

15. $\int \sin^2(x)\cos(x)dx$

16. $\int \cos^3(x)\sin(x)dx$

17. $\int \cos(3-6x)dx$

18. $\int \sec^2(4-x)dx$

19. $\int \sec(2x)dx$

20. $\int \tan^2(x)\sec^2(x)dx$

21. $\int x\cos\left(x^2\right)\,dx$

22. $\int \tan^2(x)dx$

23. $\int \cot x\,dx$. Do not just refer to Theorem 6.1.2 for the answer; justify it through Substitution.

24. $\int \csc x\,dx$. Do not just refer to Theorem 6.1.2 for the answer; justify it through Substitution.

In Exercises 25 – 32, use Substitution to evaluate the indefinite integral involving exponential functions.

25. $\int e^{3x-1}dx$

26. $\int e^{x^3}x^2dx$

27. $\int e^{x^2-2x+1}(x-1)dx$

28. $\int \frac{e^x+1}{e^x}dx$

29. $\int \frac{e^x}{e^x+1}\,dx$

30. $\int \frac{e^x-e^{-x}}{e^{2x}}dx$

31. $\int 3^{3x}dx$

32. $\int 4^{2x}dx$

In Exercises 33 – 36, use Substitution to evaluate the indefinite integral involving logarithmic functions.

33. $\int \frac{\ln x}{x}dx$

34. $\int \frac{\left(\ln x\right)^2}{x}dx$

35. $\int \frac{\ln\left(x^3\right)}{x}\,dx$

36. $\int \frac{1}{x\ln\left(x^2\right)}\,dx$

In Exercises 37 – 42, use Substitution to evaluate the indefinite integral involving rational functions.

37. $\int \frac{x^2+3x+1}{x}\,dx$

38. $\int \frac{x^3+x^2+x+1}{x}\,dx$

39. $\int \frac{x^3-1}{x+1}\,dx$

40. $\int \frac{x^2+2x-5}{x-3}\,dx$

41. $\int \frac{3x^2-5x+7}{x+1}\,dx$

42. $\int \frac{x^2+2x+1}{x^3+3x^2+3x}\,dx$

In Exercises 43 – 52, use Substitution to evaluate the indefinite integral involving inverse trigonometric functions.

43. $\int \frac{7}{x^2+7}\,dx$

44. $\int \frac{3}{\sqrt{9-x^2}}\,dx$

45. $\int \frac{14}{\sqrt{5-x^2}}\,dx$

46. $\int \frac{2}{x\sqrt{x^2-9}}\,dx$

47. $\int \frac{5}{\sqrt{x^4-16x^2}}\,dx$

48. $\int \frac{x}{\sqrt{1-x^4}}\,dx$

49. $\int \frac{1}{x^2-2x+8}\,dx$

50. $\int \frac{2}{\sqrt{-x^2+6x+7}}\,dx$

51. $\int \frac{3}{\sqrt{-x^2+8x+9}}\,dx$

52. $\int \frac{5}{x^2+6x+34}\,dx$

In Exercises 53 – 78, evaluate the indefinite integral.

53. $\int \frac{x^2}{\left(x^3+3\right)^2}\,dx$

54. $\int \left(3x^2+2x\right)\left(5x^3+5x^2+2\right)^8\,dx$

55. $\int \frac{x}{\sqrt{1-x^2}}\,dx$

56. $\int x^2\csc^2\left(x^3+1\right)\,dx$

57. $\int \sin(x)\sqrt{\cos(x)}\,dx$

58. $\int \sin\left(5x+1\right)\,dx$

59. $\int \frac{1}{x-5}\,dx$

60. $\int \frac{7}{3x+2}\,dx$

61. $\int \frac{3x^3+4x^2+2x-22}{x^2+3x+5}\,dx$

62. $\int \frac{2x+7}{x^2+7x+3}\,dx$

63. $\int \frac{9(2x+3)}{3x^2+9x+7}\,dx$

64. $\int \frac{-x^3+14x^2-46x-7}{x^2-7x+1}\,dx$

65. $\int \frac{x}{x^4+81}\,dx$

66. $\int \frac{2}{4x^2+1}\,dx$

67. $\int \frac{1}{x\sqrt{4x^2-1}}\,dx$

68. $\int \frac{1}{\sqrt{16-9x^2}}\,dx$

69. $\int \frac{3x-2}{x^2-2x+10}\,dx$

70. $\int \frac{7-2x}{x^2+12x+61}\,dx$

71. $\int \frac{x^2+5x-2}{x^2-10x+32}\,dx$

72. $\int \frac{x^3}{x^2+9}\,dx$

73. $\int \frac{x^3 - x}{x^2 + 4x + 9} dx$

74. $\int \frac{\sin(x)}{\cos^2(x) + 1} dx$

75. $\int \frac{\cos(x)}{\sin^2(x) + 1} dx$

76. $\int \frac{\cos(x)}{1 - \sin^2(x)} dx$

77. $\int \frac{3x - 3}{\sqrt{x^2 - 2x - 6}} dx$

78. $\int \frac{x - 3}{\sqrt{x^2 - 6x + 8}} dx$

In Exercises 79 – 86, evaluate the definite integral.

79. $\int_1^3 \frac{1}{x - 5} dx$

80. $\int_2^6 x\sqrt{x - 2}\,dx$

81. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos x \, dx$

82. $\int_0^1 2x(1 - x^2)^4 \, dx$

83. $\int_{-2}^{-1} (x + 1)e^{x^2 + 2x + 1} \, dx$

84. $\int_{-1}^{1} \frac{1}{1 + x^2} \, dx$

85. $\int_2^4 \frac{1}{x^2 - 6x + 10} \, dx$

86. $\int_1^{\sqrt{3}} \frac{1}{\sqrt{4 - x^2}} \, dx$

6.2 Integration by Parts

Here's a simple integral that we can't yet evaluate:

$$\int x \cos x \, dx.$$

It's a simple matter to take the derivative of the integrand using the Product Rule, but there is no Product Rule for integrals. However, this section introduces *Integration by Parts*, a method of integration that is based on the Product Rule for derivatives. It will enable us to evaluate this integral.

The Product Rule says that if u and v are functions of x, then $(uv)' = u'v + uv'$. For simplicity, we've written u for $u(x)$ and v for $v(x)$. Suppose we integrate both sides with respect to x. This gives

$$\int (uv)' \, dx = \int (u'v + uv') \, dx.$$

By the Fundamental Theorem of Calculus, the left side integrates to uv. The right side can be broken up into two integrals, and we have

$$uv = \int u'v \, dx + \int uv' \, dx.$$

Solving for the second integral we have

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Using differential notation, we can write $du = u'(x)dx$ and $dv = v'(x)dx$ and the expression above can be written as follows:

$$\int u \, dv = uv - \int v \, du.$$

This is the Integration by Parts formula. For reference purposes, we state this in a theorem.

Theorem 6.2.1 Integration by Parts

Let u and v be differentiable functions of x on an interval I containing a and b. Then

$$\int u \, dv = uv - \int v \, du,$$

and

$$\int_{x=a}^{x=b} u \, dv = uv\Big|_a^b - \int_{x=a}^{x=b} v \, du.$$

Let's try an example to understand our new technique.

Example 6.2.1 Integrating using Integration by Parts

Evaluate $\displaystyle\int x \cos x \, dx$.

Solution The key to Integration by Parts is to identify part of the integrand as "u" and part as "dv." Regular practice will help one make good identifications, and later we will introduce some principles that help. For now, let $u = x$ and $dv = \cos x \, dx$.

It is generally useful to make a small table of these values as done below. Right now we only know u and dv as shown on the left of Figure 6.2.1; on the right we fill in the rest of what we need. If $u = x$, then $du = dx$. Since $dv = \cos x\,dx$, v is an antiderivative of $\cos x$. We choose $v = \sin x$.

$$\begin{array}{ll} u = x & v = ? \\ du = ? & dv = \cos x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = x & v = \sin x \\ du = dx & dv = \cos x\,dx \end{array}$$

Figure 6.2.1: Setting up Integration by Parts.

Now substitute all of this into the Integration by Parts formula, giving

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx.$$

We can then integrate $\sin x$ to get $-\cos x + C$ and overall our answer is

$$\int x\cos x\,dx = x\sin x + \cos x + C.$$

Note how the antiderivative contains a product, $x\sin x$. This product is what makes Integration by Parts necessary.

The example above demonstrates how Integration by Parts works in general. We try to identify u and dv in the integral we are given, and the key is that we usually want to choose u and dv so that du is simpler than u and v is hopefully not too much more complicated than dv. This will mean that the integral on the right side of the Integration by Parts formula, $\int v\,du$ will be simpler to integrate than the original integral $\int u\,dv$.

In the example above, we chose $u = x$ and $dv = \cos x\,dx$. Then $du = dx$ was simpler than u and $v = \sin x$ is no more complicated than dv. Therefore, instead of integrating $x\cos x\,dx$, we could integrate $\sin x\,dx$, which we knew how to do.

A useful mnemonic for helping to determine u is "LIATE," where

L = **L**ogarithmic, I = **I**nverse Trig., A = **A**lgebraic (polynomials),
T = **T**rigonometric, and E = **E**xponential.

If the integrand contains both a logarithmic and an algebraic term, in general letting u be the logarithmic term works best, as indicated by L coming before A in LIATE.

We now consider another example.

Example 6.2.2 Integrating using Integration by Parts

Evaluate $\int xe^x\,dx$.

Solution The integrand contains an **A**lgebraic term (x) and an **E**xponential term (e^x). Our mnemonic suggests letting u be the algebraic term, so we choose $u = x$ and $dv = e^x\,dx$. Then $du = dx$ and $v = e^x$ as indicated by the tables below.

$$\begin{array}{ll} u = x & v = ? \\ du = ? & dv = e^x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = x & v = e^x \\ du = dx & dv = e^x\,dx \end{array}$$

Figure 6.2.2: Setting up Integration by Parts.

We see du is simpler than u, while there is no change in going from dv to v. This is good. The Integration by Parts formula gives

$$\int xe^x\,dx = xe^x - \int e^x\,dx.$$

The integral on the right is simple; our final answer is

$$\int xe^x\,dx = xe^x - e^x + C.$$

Note again how the antiderivatives contain a product term.

Example 6.2.3 Integrating using Integration by Parts

Evaluate $\int x^2 \cos x\,dx$.

Solution The mnemonic suggests letting $u = x^2$ instead of the trigonometric function, hence $dv = \cos x\,dx$. Then $du = 2x\,dx$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = x^2 & v = ? \\ du = ? & dv = \cos x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = x^2 & v = \sin x \\ du = 2x\,dx & dv = \cos x\,dx \end{array}$$

Figure 6.2.3: Setting up Integration by Parts.

The Integration by Parts formula gives

$$\int x^2 \cos x\,dx = x^2 \sin x - \int 2x \sin x\,dx.$$

At this point, the integral on the right is indeed simpler than the one we started with, but to evaluate it, we need to do Integration by Parts again. Here we choose $u = 2x$ and $dv = \sin x$ and fill in the rest below.

$$\begin{array}{cc} u = 2x & v = ? \\ du = ? & dv = \sin x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = 2x & v = -\cos x \\ du = 2\,dx & dv = \sin x\,dx \end{array}$$

Figure 6.2.4: Setting up Integration by Parts (again).

$$\int x^2 \cos x\,dx = x^2 \sin x - \left(-2x \cos x - \int -2\cos x\,dx \right).$$

The integral all the way on the right is now something we can evaluate. It evaluates to $-2\sin x$. Then going through and simplifying, being careful to keep all the signs straight, our answer is

$$\int x^2 \cos x\,dx = x^2 \sin x + 2x \cos x - 2\sin x + C.$$

Example 6.2.4 Integrating using Integration by Parts

Evaluate $\int e^x \cos x\,dx$.

Solution This is a classic problem. Our mnemonic suggests letting u be the trigonometric function instead of the exponential. In this particular example, one can let u be either $\cos x$ or e^x; to demonstrate that we do not have to follow LIATE, we choose $u = e^x$ and hence $dv = \cos x\,dx$. Then $du = e^x\,dx$ and $v = \sin x$ as shown below.

$$\begin{array}{cc} u = e^x & v = ? \\ du = ? & dv = \cos x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = e^x & v = \sin x \\ du = e^x\,dx & dv = \cos x\,dx \end{array}$$

Figure 6.2.5: Setting up Integration by Parts.

Notice that du is no simpler than u, going against our general rule (but bear with us). The Integration by Parts formula yields

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx.$$

The integral on the right is not much different than the one we started with, so it seems like we have gotten nowhere. Let's keep working and apply Integration by Parts to the new integral, using $u = e^x$ and $dv = \sin x\,dx$. This leads us to the following:

$$\begin{array}{cc} u = e^x & v = ? \\ du = ? & dv = \sin x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{cc} u = e^x & v = -\cos x \\ du = e^x\,dx & dv = \sin x\,dx \end{array}$$

Figure 6.2.6: Setting up Integration by Parts (again).

The Integration by Parts formula then gives:

$$\begin{aligned} \int e^x \cos x\,dx &= e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x\,dx\right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x\,dx. \end{aligned}$$

It seems we are back right where we started, as the right hand side contains $\int e^x \cos x\,dx$. But this is actually a good thing.

Add $\displaystyle\int e^x \cos x\,dx$ to both sides. This gives

$$2\int e^x \cos x\,dx = e^x \sin x + e^x \cos x$$

Now divide both sides by 2:

$$\int e^x \cos x\,dx = \frac{1}{2}\big(e^x \sin x + e^x \cos x\big).$$

Simplifying a little and adding the constant of integration, our answer is thus

$$\int e^x \cos x\,dx = \frac{1}{2}e^x\left(\sin x + \cos x\right) + C.$$

Example 6.2.5 Integrating using Integration by Parts: antiderivative of $\ln x$

Evaluate $\displaystyle\int \ln x\,dx$.

Solution One may have noticed that we have rules for integrating the familiar trigonometric functions and e^x, but we have not yet given a rule for integrating $\ln x$. That is because $\ln x$ can't easily be integrated with any of the rules we have learned up to this point. But we can find its antiderivative by a clever application of Integration by Parts. Set $u = \ln x$ and $dv = dx$. This is a good, sneaky trick to learn as it can help in other situations. This determines $du = (1/x)\,dx$ and $v = x$ as shown below.

$$\begin{array}{ll} u = \ln x & v = ? \\ du = ? & dv = dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \ln x & v = x \\ du = 1/x\,dx & dv = dx \end{array}$$

Figure 6.2.7: Setting up Integration by Parts.

Putting this all together in the Integration by Parts formula, things work out very nicely:

$$\int \ln x\,dx = x\ln x - \int x\,\frac{1}{x}\,dx.$$

The new integral simplifies to $\int 1\,dx$, which is about as simple as things get. Its integral is $x + C$ and our answer is

$$\int \ln x\,dx = x\ln x - x + C.$$

Example 6.2.6 Integrating using Int. by Parts: antiderivative of $\arctan x$

Evaluate $\displaystyle\int \arctan x\,dx$.

Solution The same sneaky trick we used above works here. Let $u = \arctan x$ and $dv = dx$. Then $du = 1/(1+x^2)\,dx$ and $v = x$. The Integration by Parts formula gives

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

The integral on the right can be solved by substitution. Taking $u = 1 + x^2$, we get $du = 2x\,dx$. The integral then becomes

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\int \frac{1}{u}\,du.$$

The integral on the right evaluates to $\frac{1}{2}\ln|u| + C$, which becomes $\frac{1}{2}\ln(1+x^2)+C$. Therefore, the answer is

$$\int \arctan x\,dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + C.$$

Substitution Before Integration

When taking derivatives, it was common to employ multiple rules (such as using both the Quotient and the Chain Rules). It should then come as no surprise that some integrals are best evaluated by combining integration techniques. In particular, here we illustrate making an "unusual" substitution first before using Integration by Parts.

Example 6.2.7 Integration by Parts after substitution

Evaluate $\int \cos(\ln x)\, dx$.

Solution The integrand contains a composition of functions, leading us to think Substitution would be beneficial. Letting $u = \ln x$, we have $du = 1/x\, dx$. This seems problematic, as we do not have a $1/x$ in the integrand. But consider:

$$du = \frac{1}{x}\, dx \Rightarrow x \cdot du = dx.$$

Since $u = \ln x$, we can use inverse functions and conclude that $x = e^u$. Therefore we have that

$$\begin{aligned} dx &= x \cdot du \\ &= e^u\, du. \end{aligned}$$

We can thus replace $\ln x$ with u and dx with $e^u\, du$. Thus we rewrite our integral as

$$\int \cos(\ln x)\, dx = \int e^u \cos u\, du.$$

We evaluated this integral in Example 6.2.4. Using the result there, we have:

$$\begin{aligned} \int \cos(\ln x)\, dx &= \int e^u \cos u\, du \\ &= \frac{1}{2}e^u\big(\sin u + \cos u\big) + C \\ &= \frac{1}{2}e^{\ln x}\big(\sin(\ln x) + \cos(\ln x)\big) + C \\ &= \frac{1}{2}x\big(\sin(\ln x) + \cos(\ln x)\big) + C. \end{aligned}$$

Definite Integrals and Integration By Parts

So far we have focused only on evaluating indefinite integrals. Of course, we can use Integration by Parts to evaluate definite integrals as well, as Theorem 6.2.1 states. We do so in the next example.

Example 6.2.8 Definite integration using Integration by Parts

Evaluate $\int_1^2 x^2 \ln x\, dx$.

Solution Our mnemonic suggests letting $u = \ln x$, hence $dv = x^2\, dx$. We then get $du = (1/x)\, dx$ and $v = x^3/3$ as shown below.

$$\begin{array}{llcll} u = \ln x & v = ? & \Rightarrow & u = \ln x & v = x^3/3 \\ du = ? & dv = x^2\,dx & & du = 1/x\,dx & dv = x^2\,dx \end{array}$$

Figure 6.2.8: Setting up Integration by Parts.

The Integration by Parts formula then gives

$$\begin{aligned} \int_1^2 x^2 \ln x\,dx &= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^3}{3}\,\frac{1}{x}\,dx \\ &= \frac{x^3}{3}\ln x\Big|_1^2 - \int_1^2 \frac{x^2}{3}\,dx \\ &= \frac{x^3}{3}\ln x\Big|_1^2 - \frac{x^3}{9}\Big|_1^2 \\ &= \left(\frac{x^3}{3}\ln x - \frac{x^3}{9}\right)\Big|_1^2 \\ &= \left(\frac{8}{3}\ln 2 - \frac{8}{9}\right) - \left(\frac{1}{3}\ln 1 - \frac{1}{9}\right) \\ &= \frac{8}{3}\ln 2 - \frac{7}{9} \\ &\approx 1.07. \end{aligned}$$

In general, Integration by Parts is useful for integrating certain products of functions, like $\int xe^x\,dx$ or $\int x^3 \sin x\,dx$. It is also useful for integrals involving logarithms and inverse trigonometric functions.

As stated before, integration is generally more difficult than derivation. We are developing tools for handling a large array of integrals, and experience will tell us when one tool is preferable/necessary over another. For instance, consider the three similar–looking integrals

$$\int xe^x\,dx, \qquad \int xe^{x^2}\,dx \qquad \text{and} \qquad \int xe^{x^3}\,dx.$$

While the first is calculated easily with Integration by Parts, the second is best approached with Substitution. Taking things one step further, the third integral has no answer in terms of elementary functions, so none of the methods we learn in calculus will get us the exact answer.

Integration by Parts is a very useful method, second only to Substitution. In the following sections of this chapter, we continue to learn other integration techniques. The next section focuses on handling integrals containing trigonometric functions.

Exercises 6.2

Terms and Concepts

1. T/F: Integration by Parts is useful in evaluating integrands that contain products of functions.

2. T/F: Integration by Parts can be thought of as the "opposite of the Chain Rule."

3. For what is "LIATE" useful?

4. T/F: If the integral that results from Integration by Parts appears to also need Integration by Parts, then a mistake was made in the orginal choice of "u".

Problems

In Exercises 5 – 34, evaluate the given indefinite integral.

5. $\int x \sin x \, dx$

6. $\int x e^{-x} \, dx$

7. $\int x^2 \sin x \, dx$

8. $\int x^3 \sin x \, dx$

9. $\int x e^{x^2} \, dx$

10. $\int x^3 e^x \, dx$

11. $\int x e^{-2x} \, dx$

12. $\int e^x \sin x \, dx$

13. $\int e^{2x} \cos x \, dx$

14. $\int e^{2x} \sin(3x) \, dx$

15. $\int e^{5x} \cos(5x) \, dx$

16. $\int \sin x \cos x \, dx$

17. $\int \sin^{-1} x \, dx$

18. $\int \tan^{-1}(2x) \, dx$

19. $\int x \tan^{-1} x \, dx$

20. $\int \sin^{-1} x \, dx$

21. $\int x \ln x \, dx$

22. $\int (x-2) \ln x \, dx$

23. $\int x \ln(x-1) \, dx$

24. $\int x \ln(x^2) \, dx$

25. $\int x^2 \ln x \, dx$

26. $\int (\ln x)^2 \, dx$

27. $\int \left(\ln(x+1)\right)^2 \, dx$

28. $\int x \sec^2 x \, dx$

29. $\int x \csc^2 x \, dx$

30. $\int x\sqrt{x-2} \, dx$

31. $\int x\sqrt{x^2-2} \, dx$

32. $\int \sec x \tan x \, dx$

33. $\int x \sec x \tan x \, dx$

34. $\int x \csc x \cot x \, dx$

In Exercises 35 – 40, evaluate the indefinite integral after first making a substitution.

35. $\int \sin(\ln x) \, dx$

36. $\int e^{2x} \cos\left(e^x\right) \, dx$

37. $\int \sin(\sqrt{x})\, dx$

38. $\int \ln(\sqrt{x})\, dx$

39. $\int e^{\sqrt{x}}\, dx$

40. $\int e^{\ln x}\, dx$

In Exercises 41 – 49, evaluate the definite integral. Note: the corresponding indefinite integrals appear in Exercises 5 – 13.

41. $\int_0^{\pi} x \sin x\, dx$

42. $\int_{-1}^{1} xe^{-x}\, dx$

43. $\int_{-\pi/4}^{\pi/4} x^2 \sin x\, dx$

44. $\int_{-\pi/2}^{\pi/2} x^3 \sin x\, dx$

45. $\int_0^{\sqrt{\ln 2}} xe^{x^2}\, dx$

46. $\int_0^1 x^3 e^x\, dx$

47. $\int_1^2 xe^{-2x}\, dx$

48. $\int_0^{\pi} e^x \sin x\, dx$

49. $\int_{-\pi/2}^{\pi/2} e^{2x} \cos x\, dx$

6.3 Trigonometric Integrals

Functions involving trigonometric functions are useful as they are good at describing periodic behaviour. This section describes several techniques for finding antiderivatives of certain combinations of trigonometric functions.

Integrals of the form $\int \sin^m x \cos^n x\, dx$

In learning the technique of Substitution, we saw the integral $\int \sin x \cos x\, dx$ in Example 6.1.4. The integration was not difficult, and one could easily evaluate the indefinite integral by letting $u = \sin x$ or by letting $u = \cos x$. This integral is easy since the power of both sine and cosine is 1.

We generalize this integral and consider integrals of the form $\int \sin^m x \cos^n x\, dx$, where m, n are nonnegative integers. Our strategy for evaluating these integrals is to use the identity $\cos^2 x + \sin^2 x = 1$ to convert high powers of one trigonometric function into the other, leaving a single sine or cosine term in the integrand. We summarize the general technique in the following Key Idea.

Key Idea 6.3.1 Integrals Involving Powers of Sine and Cosine

Consider $\int \sin^m x \cos^n x\, dx$, where m, n are nonnegative integers.

1. If m is odd, then $m = 2k+1$ for some integer k. Rewrite

   $$\sin^m x = \sin^{2k+1} x = \sin^{2k} x \sin x = (\sin^2 x)^k \sin x = (1-\cos^2 x)^k \sin x.$$

   Then

   $$\int \sin^m x \cos^n x\, dx = \int (1-\cos^2 x)^k \sin x \cos^n x\, dx = -\int (1-u^2)^k u^n\, du,$$

   where $u = \cos x$ and $du = -\sin x\, dx$.

2. If n is odd, then using substitutions similar to that outlined above we have

   $$\int \sin^m x \cos^n x\, dx = \int u^m (1-u^2)^k\, du,$$

   where $u = \sin x$ and $du = \cos x\, dx$.

3. If both m and n are even, use the power–reducing identities

   $$\cos^2 x = \frac{1+\cos(2x)}{2} \quad \text{and} \quad \sin^2 x = \frac{1-\cos(2x)}{2}$$

   to reduce the degree of the integrand. Expand the result and apply the principles of this Key Idea again.

We practice applying Key Idea 6.3.1 in the next examples.

Example 6.3.1 Integrating powers of sine and cosine

Evaluate $\int \sin^5 x \cos^8 x\, dx$.

Solution The power of the sine term is odd, so we rewrite $\sin^5 x$ as

$$\sin^5 x = \sin^4 x \sin x = (\sin^2 x)^2 \sin x = (1-\cos^2 x)^2 \sin x.$$

Our integral is now $\int (1-\cos^2 x)^2 \cos^8 x \sin x\, dx$. Let $u = \cos x$, hence $du =$

$-\sin x\,dx$. Making the substitution and expanding the integrand gives

$$\int (1-\cos^2)^2\cos^8 x\sin x\,dx = -\int (1-u^2)^2u^8\,du = -\int \big(1-2u^2+u^4\big)u^8\,du$$
$$= -\int \big(u^8-2u^{10}+u^{12}\big)\,du.$$

This final integral is not difficult to evaluate, giving

$$-\int \big(u^8-2u^{10}+u^{12}\big)\,du = -\frac{1}{9}u^9+\frac{2}{11}u^{11}-\frac{1}{13}u^{13}+C$$
$$= -\frac{1}{9}\cos^9 x+\frac{2}{11}\cos^{11}x-\frac{1}{13}\cos^{13}x+C.$$

Example 6.3.2 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \sin^5 x\cos^9 x\,dx$.

Solution The powers of both the sine and cosine terms are odd, therefore we can apply the techniques of Key Idea 6.3.1 to either power. We choose to work with the power of the cosine term since the previous example used the sine term's power.

We rewrite $\cos^9 x$ as

$$\begin{aligned}\cos^9 x &= \cos^8 x\cos x\\ &= (\cos^2 x)^4\cos x\\ &= (1-\sin^2 x)^4\cos x\\ &= (1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x)\cos x.\end{aligned}$$

We rewrite the integral as

$$\int \sin^5 x\cos^9 x\,dx = \int \sin^5 x\big(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\big)\cos x\,dx.$$

Now substitute and integrate, using $u=\sin x$ and $du=\cos x\,dx$.

$$\int \sin^5 x\big(1-4\sin^2 x+6\sin^4 x-4\sin^6 x+\sin^8 x\big)\cos x\,dx =$$
$$\int u^5(1-4u^2+6u^4-4u^6+u^8)\,du = \int \big(u^5-4u^7+6u^9-4u^{11}+u^{13}\big)\,du$$
$$= \frac{1}{6}u^6-\frac{1}{2}u^8+\frac{3}{5}u^{10}-\frac{1}{3}u^{12}+\frac{1}{14}u^{14}+C$$
$$= \frac{1}{6}\sin^6 x-\frac{1}{2}\sin^8 x+\frac{3}{5}\sin^{10} x+\ldots$$
$$-\frac{1}{3}\sin^{12} x+\frac{1}{14}\sin^{14} x+C.$$

Technology Note: The work we are doing here can be a bit tedious, but the skills developed (problem solving, algebraic manipulation, etc.) are important. Nowadays problems of this sort are often solved using a computer algebra system. The powerful program *Mathematica*® integrates $\int \sin^5 x\cos^9 x\,dx$ as

$$f(x) = -\frac{45\cos(2x)}{16384}-\frac{5\cos(4x)}{8192}+\frac{19\cos(6x)}{49152}+\frac{\cos(8x)}{4096}-\frac{\cos(10x)}{81920}-\frac{\cos(12x)}{24576}-\frac{\cos(14x)}{114688},$$

which clearly has a different form than our answer in Example 6.3.2, which is

$$g(x) = \frac{1}{6}\sin^6 x-\frac{1}{2}\sin^8 x+\frac{3}{5}\sin^{10} x-\frac{1}{3}\sin^{12} x+\frac{1}{14}\sin^{14} x.$$

Figure 6.3.1 shows a graph of f and g; they are clearly not equal, but they differ *only by a constant*. That is $g(x) = f(x) + C$ for some constant C. So we have two different antiderivatives of the same function, meaning both answers are correct.

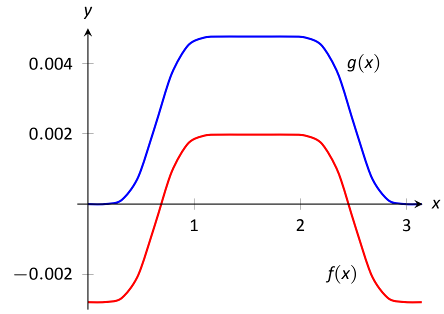


Figure 6.3.1: A plot of $f(x)$ and $g(x)$ from Example 6.3.2 and the Technology Note.

Example 6.3.3 Integrating powers of sine and cosine

Evaluate $\displaystyle\int \cos^4 x \sin^2 x\, dx$.

Solution The powers of sine and cosine are both even, so we employ the power–reducing formulas and algebra as follows.

$$\begin{aligned}\int \cos^4 x \sin^2 x\, dx &= \int \left(\frac{1+\cos(2x)}{2}\right)^2 \left(\frac{1-\cos(2x)}{2}\right) dx\\ &= \int \frac{1+2\cos(2x)+\cos^2(2x)}{4}\cdot\frac{1-\cos(2x)}{2}\, dx\\ &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\, dx\end{aligned}$$

The $\cos(2x)$ term is easy to integrate, especially with Key Idea 6.1.1. The $\cos^2(2x)$ term is another trigonometric integral with an even power, requiring the power–reducing formula again. The $\cos^3(2x)$ term is a cosine function with an odd power, requiring a substitution as done before. We integrate each in turn below.

$$\int \cos(2x)\, dx = \frac{1}{2}\sin(2x) + C.$$

$$\int \cos^2(2x)\, dx = \int \frac{1+\cos(4x)}{2}\, dx = \frac{1}{2}\big(x + \frac{1}{4}\sin(4x)\big) + C.$$

Finally, we rewrite $\cos^3(2x)$ as

$$\cos^3(2x) = \cos^2(2x)\cos(2x) = \big(1-\sin^2(2x)\big)\cos(2x).$$

Letting $u = \sin(2x)$, we have $du = 2\cos(2x)\, dx$, hence

$$\begin{aligned}\int \cos^3(2x)\, dx &= \int \big(1-\sin^2(2x)\big)\cos(2x)\, dx\\ &= \int \frac{1}{2}(1-u^2)\, du\\ &= \frac{1}{2}\left(u - \frac{1}{3}u^3\right) + C\\ &= \frac{1}{2}\left(\sin(2x) - \frac{1}{3}\sin^3(2x)\right) + C\end{aligned}$$

Putting all the pieces together, we have

$$\begin{aligned}\int \cos^4 x \sin^2 x\, dx &= \int \frac{1}{8}\big(1+\cos(2x)-\cos^2(2x)-\cos^3(2x)\big)\, dx\\ &= \frac{1}{8}\left[x + \frac{1}{2}\sin(2x) - \frac{1}{2}\big(x+\frac{1}{4}\sin(4x)\big) - \frac{1}{2}\left(\sin(2x)-\frac{1}{3}\sin^3(2x)\right)\right] + C\\ &= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin(4x) + \frac{1}{6}\sin^3(2x)\right] + C\end{aligned}$$

The process above was a bit long and tedious, but being able to work a problem such as this from start to finish is important.

Integrals of the form $\int \sin(mx)\sin(nx)\,dx$, $\int \cos(mx)\cos(nx)\,dx$, and $\int \sin(mx)\cos(nx)\,dx$.

Functions that contain products of sines and cosines of differing periods are important in many applications including the analysis of sound waves. Integrals of the form

$$\int \sin(mx)\sin(nx)\,dx, \quad \int \cos(mx)\cos(nx)\,dx \quad \text{and} \quad \int \sin(mx)\cos(nx)\,dx$$

are best approached by first applying the Product to Sum Formulas found in the back cover of this text, namely

$$\begin{aligned}
\sin(mx)\sin(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) - \cos\big((m+n)x\big)\Big] \\
\cos(mx)\cos(nx) &= \frac{1}{2}\Big[\cos\big((m-n)x\big) + \cos\big((m+n)x\big)\Big] \\
\sin(mx)\cos(nx) &= \frac{1}{2}\Big[\sin\big((m-n)x\big) + \sin\big((m+n)x\big)\Big]
\end{aligned}$$

Example 6.3.4 Integrating products of $\sin(mx)$ and $\cos(nx)$

Evaluate $\int \sin(5x)\cos(2x)\,dx$.

Solution The application of the formula and subsequent integration are straightforward:

$$\begin{aligned}
\int \sin(5x)\cos(2x)\,dx &= \int \frac{1}{2}\Big[\sin(3x) + \sin(7x)\Big]\,dx \\
&= -\frac{1}{6}\cos(3x) - \frac{1}{14}\cos(7x) + C
\end{aligned}$$

Integrals of the form $\int \tan^m x \sec^n x\,dx$.

When evaluating integrals of the form $\int \sin^m x\cos^n x\,dx$, the Pythagorean Theorem allowed us to convert even powers of sine into even powers of cosine, and vice–versa. If, for instance, the power of sine was odd, we pulled out one $\sin x$ and converted the remaining even power of $\sin x$ into a function using powers of $\cos x$, leading to an easy substitution.

The same basic strategy applies to integrals of the form $\int \tan^m x\sec^n x\,dx$, albeit a bit more nuanced. The following three facts will prove useful:

- $\frac{d}{dx}(\tan x) = \sec^2 x$,
- $\frac{d}{dx}(\sec x) = \sec x\tan x$, and
- $1 + \tan^2 x = \sec^2 x$ (the Pythagorean Theorem).

If the integrand can be manipulated to separate a $\sec^2 x$ term with the remaining secant power even, or if a $\sec x\tan x$ term can be separated with the remaining $\tan x$ power even, the Pythagorean Theorem can be employed, leading to a simple substitution. This strategy is outlined in the following Key Idea.

Key Idea 6.3.2 Integrals Involving Powers of Tangent and Secant

Consider $\int \tan^m x \sec^n x\, dx$, where m, n are nonnegative integers.

1. If n is even, then $n = 2k$ for some integer k. Rewrite $\sec^n x$ as

$$\sec^n x = \sec^{2k} x = \sec^{2k-2} x \sec^2 x = (1 + \tan^2 x)^{k-1} \sec^2 x.$$

Then

$$\int \tan^m x \sec^n x\, dx = \int \tan^m x(1 + \tan^2 x)^{k-1} \sec^2 x\, dx = \int u^m(1 + u^2)^{k-1}\, du,$$

where $u = \tan x$ and $du = \sec^2 x\, dx$.

2. If m is odd, then $m = 2k + 1$ for some integer k. Rewrite $\tan^m x \sec^n x$ as

$$\begin{aligned}\tan^m x \sec^n x &= \tan^{2k+1} x \sec^n x = \tan^{2k} x \sec^{n-1} x \sec x \tan x\\ &= (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x.\end{aligned}$$

Then

$$\int \tan^m x \sec^n x\, dx = \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x\, dx = \int (u^2 - 1)^k u^{n-1}\, du,$$

where $u = \sec x$ and $du = \sec x \tan x\, dx$.

3. If n is odd and m is even, then $m = 2k$ for some integer k. Convert $\tan^m x$ to $(\sec^2 x - 1)^k$. Expand the new integrand and use Integration By Parts, with $dv = \sec^2 x\, dx$.

4. If m is even and $n = 0$, rewrite $\tan^m x$ as

$$\tan^m x = \tan^{m-2} x \tan^2 x = \tan^{m-2} x(\sec^2 x - 1) = \tan^{m-2} \sec^2 x - \tan^{m-2} x.$$

So

$$\int \tan^m x\, dx = \underbrace{\int \tan^{m-2} \sec^2 x\, dx}_{\text{apply rule \#1}} - \underbrace{\int \tan^{m-2} x\, dx}_{\text{apply rule \#4 again}}.$$

The techniques described in items 1 and 2 of Key Idea 6.3.2 are relatively straightforward, but the techniques in items 3 and 4 can be rather tedious. A few examples will help with these methods.

Example 6.3.5 Integrating powers of tangent and secant

Evaluate $\int \tan^2 x \sec^6 x\, dx$.

Solution Since the power of secant is even, we use rule #1 from Key Idea 6.3.2 and pull out a $\sec^2 x$ in the integrand. We convert the remaining powers of secant into powers of tangent.

$$\begin{aligned}\int \tan^2 x \sec^6 x\, dx &= \int \tan^2 x \sec^4 x \sec^2 x\, dx\\ &= \int \tan^2 x\big(1 + \tan^2 x\big)^2 \sec^2 x\, dx\end{aligned}$$

Now substitute, with $u = \tan x$, with $du = \sec^2 x\,dx$.

$$= \int u^2\left(1+u^2\right)^2\,du$$

We leave the integration and subsequent substitution to the reader. The final answer is

$$= \frac{1}{3}\tan^3 x + \frac{2}{5}\tan^5 x + \frac{1}{7}\tan^7 x + C.$$

Example 6.3.6 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \sec^3 x\,dx$.

Solution We apply rule #3 from Key Idea 6.3.2 as the power of secant is odd and the power of tangent is even (0 is an even number). We use Integration by Parts; the rule suggests letting $dv = \sec^2 x\,dx$, meaning that $u = \sec x$.

$$\begin{array}{ll} u = \sec x & v = ? \\ du = ? & dv = \sec^2 x\,dx \end{array} \quad \Rightarrow \quad \begin{array}{ll} u = \sec x & v = \tan x \\ du = \sec x \tan x\,dx & dv = \sec^2 x\,dx \end{array}$$

Figure 6.3.2: Setting up Integration by Parts.

Employing Integration by Parts, we have

$$\begin{aligned}\int \sec^3 x\,dx &= \int \underbrace{\sec x}_{u} \cdot \underbrace{\sec^2 x\,dx}_{dv} \\ &= \sec x \tan x - \int \sec x \tan^2 x\,dx.\end{aligned}$$

This new integral also requires applying rule #3 of Key Idea 6.3.2:

$$\begin{aligned} &= \sec x \tan x - \int \sec x\big(\sec^2 x - 1\big)\,dx \\ &= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx \\ &= \sec x \tan x - \int \sec^3 x\,dx + \ln|\sec x + \tan x|\end{aligned}$$

In previous applications of Integration by Parts, we have seen where the original integral has reappeared in our work. We resolve this by adding $\int \sec^3 x\,dx$ to both sides, giving:

$$\begin{aligned} 2\int \sec^3 x\,dx &= \sec x \tan x + \ln|\sec x + \tan x| \\ \int \sec^3 x\,dx &= \frac{1}{2}\Big(\sec x \tan x + \ln|\sec x + \tan x|\Big) + C\end{aligned}$$

We give one more example.

Example 6.3.7 Integrating powers of tangent and secant

Evaluate $\displaystyle\int \tan^6 x\,dx$.

Solution We employ rule #4 of Key Idea 6.3.2.

$$\begin{aligned}\int \tan^6 x\,dx &= \int \tan^4 x \tan^2 x\,dx\\ &= \int \tan^4 x\big(\sec^2 x - 1\big)\,dx\\ &= \int \tan^4 x \sec^2 x\,dx - \int \tan^4 x\,dx\end{aligned}$$

Integrate the first integral with substitution, $u = \tan x$; integrate the second by employing rule #4 again.

$$\begin{aligned}&= \frac{1}{5}\tan^5 x - \int \tan^2 x \tan^2 x\,dx\\ &= \frac{1}{5}\tan^5 x - \int \tan^2 x\big(\sec^2 x - 1\big)\,dx\\ &= \frac{1}{5}\tan^5 x - \int \tan^2 x \sec^2 x\,dx + \int \tan^2 x\,dx\end{aligned}$$

Again, use substitution for the first integral and rule #4 for the second.

$$\begin{aligned}&= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \int \big(\sec^2 x - 1\big)\,dx\\ &= \frac{1}{5}\tan^5 x - \frac{1}{3}\tan^3 x + \tan x - x + C.\end{aligned}$$

These latter examples were admittedly long, with repeated applications of the same rule. Try to not be overwhelmed by the length of the problem, but rather admire how robust this solution method is. A trigonometric function of a high power can be systematically reduced to trigonometric functions of lower powers until all antiderivatives can be computed.

The next section introduces an integration technique known as Trigonometric Substitution, a clever combination of Substitution and the Pythagorean Theorem.

Exercises 6.3

Terms and Concepts

1. T/F: $\int \sin^2 x \cos^2 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are even.

2. T/F: $\int \sin^3 x \cos^3 x \, dx$ cannot be evaluated using the techniques described in this section since both powers of $\sin x$ and $\cos x$ are odd.

3. T/F: This section addresses how to evaluate indefinite integrals such as $\int \sin^5 x \tan^3 x \, dx$.

4. T/F: Sometimes computer programs evaluate integrals involving trigonometric functions differently than one would using the techniques of this section. When this is the case, the techniques of this section have failed and one should only trust the answer given by the computer.

Problems

In Exercises 5 – 28, evaluate the indefinite integral.

5. $\int \sin x \cos^4 x \, dx$

6. $\int \sin^3 x \cos x \, dx$

7. $\int \sin^3 x \cos^2 x \, dx$

8. $\int \sin^3 x \cos^3 x \, dx$

9. $\int \sin^6 x \cos^5 x \, dx$

10. $\int \sin^2 x \cos^7 x \, dx$

11. $\int \sin^2 x \cos^2 x \, dx$

12. $\int \sin x \cos x \, dx$

13. $\int \sin(5x) \cos(3x) \, dx$

14. $\int \sin(x) \cos(2x) \, dx$

15. $\int \sin(3x) \sin(7x) \, dx$

16. $\int \sin(\pi x) \sin(2\pi x) \, dx$

17. $\int \cos(x) \cos(2x) \, dx$

18. $\int \cos\left(\frac{\pi}{2}x\right) \cos(\pi x) \, dx$

19. $\int \tan^4 x \sec^2 x \, dx$

20. $\int \tan^2 x \sec^4 x \, dx$

21. $\int \tan^3 x \sec^4 x \, dx$

22. $\int \tan^3 x \sec^2 x \, dx$

23. $\int \tan^3 x \sec^3 x \, dx$

24. $\int \tan^5 x \sec^5 x \, dx$

25. $\int \tan^4 x \, dx$

26. $\int \sec^5 x \, dx$

27. $\int \tan^2 x \sec x \, dx$

28. $\int \tan^2 x \sec^3 x \, dx$

In Exercises 29 – 35, evaluate the definite integral. Note: the corresponding indefinite integrals appear in the previous set.

29. $\int_0^{\pi} \sin x \cos^4 x \, dx$

30. $\int_{-\pi}^{\pi} \sin^3 x \cos x \, dx$

31. $\int_{-\pi/2}^{\pi/2} \sin^2 x \cos^7 x \, dx$

32. $\int_0^{\pi/2} \sin(5x) \cos(3x) \, dx$

33. $\int_{-\pi/2}^{\pi/2} \cos(x) \cos(2x) \, dx$

34. $\int_0^{\pi/4} \tan^4 x \sec^2 x \, dx$

35. $\int_{-\pi/4}^{\pi/4} \tan^2 x \sec^4 x \, dx$

6.4 Trigonometric Substitution

In Section 5.2 we defined the definite integral as the "signed area under the curve." In that section we had not yet learned the Fundamental Theorem of Calculus, so we only evaluated special definite integrals which described nice, geometric shapes. For instance, we were able to evaluate

$$\int_{-3}^{3} \sqrt{9-x^2}\,dx = \frac{9\pi}{2} \tag{6.1}$$

as we recognized that $f(x) = \sqrt{9-x^2}$ described the upper half of a circle with radius 3.

We have since learned a number of integration techniques, including Substitution and Integration by Parts, yet we are still unable to evaluate the above integral without resorting to a geometric interpretation. This section introduces Trigonometric Substitution, a method of integration that fills this gap in our integration skill. This technique works on the same principle as Substitution as found in Section 6.1, though it can feel "backward." In Section 6.1, we set $u = f(x)$, for some function f, and replaced $f(x)$ with u. In this section, we will set $x = f(\theta)$, where f is a trigonometric function, then replace x with $f(\theta)$.

We start by demonstrating this method in evaluating the integral in Equation (6.1). After the example, we will generalize the method and give more examples.

Example 6.4.1 Using Trigonometric Substitution

Evaluate $\displaystyle\int_{-3}^{3} \sqrt{9-x^2}\,dx$.

SOLUTION We begin by noting that $9\sin^2\theta + 9\cos^2\theta = 9$, and hence $9\cos^2\theta = 9-9\sin^2\theta$. If we let $x = 3\sin\theta$, then $9-x^2 = 9-9\sin^2\theta = 9\cos^2\theta$.

Setting $x = 3\sin\theta$ gives $dx = 3\cos\theta\,d\theta$. We are almost ready to substitute. We also wish to change our bounds of integration. The bound $x = -3$ corresponds to $\theta = -\pi/2$ (for when $\theta = -\pi/2$, $x = 3\sin\theta = -3$). Likewise, the bound of $x = 3$ is replaced by the bound $\theta = \pi/2$. Thus

$$\begin{aligned}
\int_{-3}^{3} \sqrt{9-x^2}\,dx &= \int_{-\pi/2}^{\pi/2} \sqrt{9-9\sin^2\theta}(3\cos\theta)\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3\sqrt{9\cos^2\theta}\cos\theta\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} 3|3\cos\theta|\cos\theta\,d\theta.
\end{aligned}$$

On $[-\pi/2, \pi/2]$, $\cos\theta$ is always positive, so we can drop the absolute value bars, then employ a power–reducing formula:

$$\begin{aligned}
&= \int_{-\pi/2}^{\pi/2} 9\cos^2\theta\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} \frac{9}{2}\big(1+\cos(2\theta)\big)\,d\theta \\
&= \frac{9}{2}\big(\theta + \frac{1}{2}\sin(2\theta)\big)\Big|_{-\pi/2}^{\pi/2} = \frac{9}{2}\pi.
\end{aligned}$$

This matches our answer from before.

We now describe in detail Trigonometric Substitution. This method excels when dealing with integrands that contain $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ and $\sqrt{x^2+a^2}$. The following Key Idea outlines the procedure for each case, followed by more examples. Each right triangle acts as a reference to help us understand the relationships between x and θ.

Key Idea 6.4.1 Trigonometric Substitution

For the three cases below, we assume that $a > 0$.

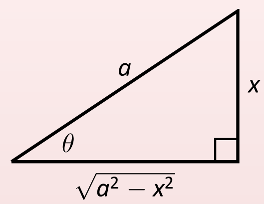
(a) For integrands containing $\sqrt{a^2-x^2}$:

Let $x = a\sin\theta$, $dx = a\cos\theta\,d\theta$

Thus $\theta = \sin^{-1}(x/a)$, for $-\pi/2 \le \theta \le \pi/2$.

On this interval, $\cos\theta \ge 0$, so

$\sqrt{a^2-x^2} = a\cos\theta$



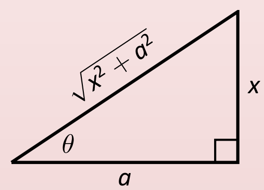
(b) For integrands containing $\sqrt{x^2+a^2}$:

Let $x = a\tan\theta$, $dx = a\sec^2\theta\,d\theta$

Thus $\theta = \tan^{-1}(x/a)$, for $-\pi/2 < \theta < \pi/2$.

On this interval, $\sec\theta > 0$, so

$\sqrt{x^2+a^2} = a\sec\theta$



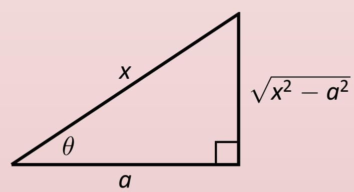
(c) For integrands containing $\sqrt{x^2-a^2}$:

Let $x = a\sec\theta$, $dx = a\sec\theta\tan\theta\,d\theta$

Thus $\theta = \sec^{-1}(x/a)$. Note that $\sqrt{x^2-a^2}$ is defined for $x \ge a$ or $x \le -a$.
If $x \ge a$, then $x/a \ge 1$ and $0 \le \theta < \pi/2$; if $x < -a$, then $x/a \le -1$ and $\pi \le \theta < \frac{3\pi}{2}$.

On these intervals, $\tan\theta \ge 0$, so

$\sqrt{x^2-a^2} = a\tan\theta$



Example 6.4.2 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{\sqrt{5+x^2}}\,dx$.

Solution Using Key Idea 6.4.1(b), we recognize $a = \sqrt{5}$ and set $x = \sqrt{5}\tan\theta$. This makes $dx = \sqrt{5}\sec^2\theta\,d\theta$. We will use the fact that $\sqrt{5+x^2} = \sqrt{5+5\tan^2\theta} = \sqrt{5\sec^2\theta} = \sqrt{5}\sec\theta$. Substituting, we have:

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \int \frac{1}{\sqrt{5+5\tan^2\theta}}\sqrt{5}\sec^2\theta\,d\theta\\ &= \int \frac{\sqrt{5}\sec^2\theta}{\sqrt{5}\sec\theta}\,d\theta\\ &= \int \sec\theta\,d\theta\\ &= \ln\big|\sec\theta+\tan\theta\big| + C.\end{aligned}$$

While the integration steps are over, we are not yet done. The original problem was stated in terms of x, whereas our answer is given in terms of θ. We must convert back to x.

The reference triangle given in Key Idea 6.4.1(b) helps. With $x = \sqrt{5}\tan\theta$, we have

$$\tan\theta = \frac{x}{\sqrt{5}} \quad \text{and} \quad \sec\theta = \frac{\sqrt{x^2+5}}{\sqrt{5}}.$$

This gives

$$\begin{aligned}\int \frac{1}{\sqrt{5+x^2}}\,dx &= \ln\left|\sec\theta + \tan\theta\right| + C \\ &= \ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right| + C.\end{aligned}$$

We can leave this answer as is, or we can use a logarithmic identity to simplify it. Note:

$$\begin{aligned}\ln\left|\frac{\sqrt{x^2+5}}{\sqrt{5}} + \frac{x}{\sqrt{5}}\right| + C &= \ln\left|\frac{1}{\sqrt{5}}\left(\sqrt{x^2+5}+x\right)\right| + C \\ &= \ln\left|\frac{1}{\sqrt{5}}\right| + \ln\left|\sqrt{x^2+5}+x\right| + C \\ &= \ln\left|\sqrt{x^2+5}+x\right| + C,\end{aligned}$$

where the $\ln\left(1/\sqrt{5}\right)$ term is absorbed into the constant C. (In Section 6.6 we will learn another way of approaching this problem.)

Example 6.4.3 Using Trigonometric Substitution

Evaluate $\int \sqrt{4x^2-1}\,dx$.

Solution We start by rewriting the integrand so that it looks like $\sqrt{x^2-a^2}$ for some value of a:

$$\begin{aligned}\sqrt{4x^2-1} &= \sqrt{4\left(x^2-\frac{1}{4}\right)} \\ &= 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}.\end{aligned}$$

So we have $a = 1/2$, and following Key Idea 6.4.1(c), we set $x = \frac{1}{2}\sec\theta$, and hence $dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$. We now rewrite the integral with these substitutions:

$$\begin{aligned}
\int \sqrt{4x^2-1}\,dx &= \int 2\sqrt{x^2-\left(\frac{1}{2}\right)^2}\,dx \\
&= \int 2\sqrt{\frac{1}{4}\sec^2\theta-\frac{1}{4}}\left(\frac{1}{2}\sec\theta\tan\theta\right)\,d\theta \\
&= \int \sqrt{\frac{1}{4}(\sec^2\theta-1)}\Big(\sec\theta\tan\theta\Big)\,d\theta \\
&= \int \sqrt{\frac{1}{4}\tan^2\theta}\Big(\sec\theta\tan\theta\Big)\,d\theta \\
&= \int \frac{1}{2}\tan^2\theta\sec\theta\,d\theta \\
&= \frac{1}{2}\int \Big(\sec^2\theta-1\Big)\sec\theta\,d\theta \\
&= \frac{1}{2}\int \big(\sec^3\theta-\sec\theta\big)\,d\theta.
\end{aligned}$$

We integrated $\sec^3\theta$ in Example 6.3.6, finding its antiderivatives to be

$$\int \sec^3\theta\,d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)+C.$$

Thus

$$\begin{aligned}
\int \sqrt{4x^2-1}\,dx &= \frac{1}{2}\int \big(\sec^3\theta-\sec\theta\big)\,d\theta \\
&= \frac{1}{2}\left(\frac{1}{2}\Big(\sec\theta\tan\theta+\ln|\sec\theta+\tan\theta|\Big)-\ln|\sec\theta+\tan\theta|\right)+C \\
&= \frac{1}{4}\left(\sec\theta\tan\theta-\ln|\sec\theta+\tan\theta|\right)+C.
\end{aligned}$$

We are not yet done. Our original integral is given in terms of x, whereas our final answer, as given, is in terms of θ. We need to rewrite our answer in terms of x. With $a=1/2$, and $x=\frac{1}{2}\sec\theta$, the reference triangle in Key Idea 6.4.1(c) shows that

$$\tan\theta=\sqrt{x^2-1/4}\Big/(1/2)=2\sqrt{x^2-1/4}\quad\text{and}\quad\sec\theta=2x.$$

Thus

$$\begin{aligned}
\frac{1}{4}\Big(\sec\theta\tan\theta-\ln\big|\sec\theta+\tan\theta\big|\Big)+C &= \frac{1}{4}\Big(2x\cdot2\sqrt{x^2-1/4}-\ln\big|2x+2\sqrt{x^2-1/4}\big|\Big)+C \\
&= \frac{1}{4}\Big(4x\sqrt{x^2-1/4}-\ln\big|2x+2\sqrt{x^2-1/4}\big|\Big)+C.
\end{aligned}$$

The final answer is given in the last line above, repeated here:

$$\int \sqrt{4x^2-1}\,dx=\frac{1}{4}\Big(4x\sqrt{x^2-1/4}-\ln\big|2x+2\sqrt{x^2-1/4}\big|\Big)+C.$$

Example 6.4.4 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{\sqrt{4-x^2}}{x^2}\,dx$.

Solution We use Key Idea 6.4.1(a) with $a=2$, $x=2\sin\theta$, $dx=$

$2\cos\theta$ and hence $\sqrt{4-x^2} = 2\cos\theta$. This gives

$$\begin{aligned}\int \frac{\sqrt{4-x^2}}{x^2}\,dx &= \int \frac{2\cos\theta}{4\sin^2\theta}(2\cos\theta)\,d\theta\\ &= \int \cot^2\theta\,d\theta\\ &= \int (\csc^2\theta - 1)\,d\theta\\ &= -\cot\theta - \theta + C.\end{aligned}$$

We need to rewrite our answer in terms of x. Using the reference triangle found in Key Idea 6.4.1(a), we have $\cot\theta = \sqrt{4-x^2}/x$ and $\theta = \sin^{-1}(x/2)$. Thus

$$\int \frac{\sqrt{4-x^2}}{x^2}\,dx = -\frac{\sqrt{4-x^2}}{x} - \sin^{-1}\left(\frac{x}{2}\right) + C.$$

Trigonometric Substitution can be applied in many situations, even those not of the form $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$ or $\sqrt{x^2+a^2}$. In the following example, we apply it to an integral we already know how to handle.

Example 6.4.5 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{x^2+1}\,dx$.

Solution We know the answer already as $\tan^{-1}x + C$. We apply Trigonometric Substitution here to show that we get the same answer without inherently relying on knowledge of the derivative of the arctangent function.

Using Key Idea 6.4.1(b), let $x = \tan\theta$, $dx = \sec^2\theta\,d\theta$ and note that $x^2+1 = \tan^2\theta + 1 = \sec^2\theta$. Thus

$$\begin{aligned}\int \frac{1}{x^2+1}\,dx &= \int \frac{1}{\sec^2\theta}\sec^2\theta\,d\theta\\ &= \int 1\,d\theta\\ &= \theta + C.\end{aligned}$$

Since $x = \tan\theta$, $\theta = \tan^{-1}x$, and we conclude that $\displaystyle\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$.

The next example is similar to the previous one in that it does not involve a square–root. It shows how several techniques and identities can be combined to obtain a solution.

Example 6.4.6 Using Trigonometric Substitution

Evaluate $\displaystyle\int \frac{1}{(x^2+6x+10)^2}\,dx$.

Solution We start by completing the square, then make the substitution $u = x+3$, followed by the trigonometric substitution of $u = \tan\theta$:

$$\int \frac{1}{(x^2+6x+10)^2}\,dx = \int \frac{1}{\big((x+3)^2+1\big)^2}\,dx = \int \frac{1}{(u^2+1)^2}\,du.$$

Now make the substitution $u = \tan\theta$, $du = \sec^2\theta\, d\theta$:

$$\begin{aligned} &= \int \frac{1}{(\tan^2\theta + 1)^2} \sec^2\theta\, d\theta \\ &= \int \frac{1}{(\sec^2\theta)^2} \sec^2\theta\, d\theta \\ &= \int \cos^2\theta\, d\theta. \end{aligned}$$

Applying a power reducing formula, we have

$$\begin{aligned} &= \int \left(\frac{1}{2} + \frac{1}{2}\cos(2\theta)\right) d\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C. \quad (6.2) \end{aligned}$$

We need to return to the variable x. As $u = \tan\theta$, $\theta = \tan^{-1} u$. Using the identity $\sin(2\theta) = 2\sin\theta\cos\theta$ and using the reference triangle found in Key Idea 6.4.1(b), we have

$$\frac{1}{4}\sin(2\theta) = \frac{1}{2}\frac{u}{\sqrt{u^2+1}} \cdot \frac{1}{\sqrt{u^2+1}} = \frac{1}{2}\frac{u}{u^2+1}.$$

Finally, we return to x with the substitution $u = x + 3$. We start with the expression in Equation (6.2):

$$\begin{aligned} \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C &= \frac{1}{2}\tan^{-1} u + \frac{1}{2}\frac{u}{u^2+1} + C \\ &= \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C. \end{aligned}$$

Stating our final result in one line,

$$\int \frac{1}{(x^2+6x+10)^2}\, dx = \frac{1}{2}\tan^{-1}(x+3) + \frac{x+3}{2(x^2+6x+10)} + C.$$

Our last example returns us to definite integrals, as seen in our first example. Given a definite integral that can be evaluated using Trigonometric Substitution, we could first evaluate the corresponding indefinite integral (by changing from an integral in terms of x to one in terms of θ, then converting back to x) and then evaluate using the original bounds. It is much more straightforward, though, to change the bounds as we substitute.

Example 6.4.7 Definite integration and Trigonometric Substitution

Evaluate $\displaystyle\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\, dx.$

Solution Using Key Idea 6.4.1(b), we set $x = 5\tan\theta$, $dx = 5\sec^2\theta\, d\theta$, and note that $\sqrt{x^2+25} = 5\sec\theta$. As we substitute, we can also change the bounds of integration.

The lower bound of the original integral is $x = 0$. As $x = 5\tan\theta$, we solve for θ and find $\theta = \tan^{-1}(x/5)$. Thus the new lower bound is $\theta = \tan^{-1}(0) = 0$. The original upper bound is $x = 5$, thus the new upper bound is $\theta = \tan^{-1}(5/5) = \pi/4$.

Thus we have

$$\int_0^5 \frac{x^2}{\sqrt{x^2+25}}\,dx = \int_0^{\pi/4} \frac{25\tan^2\theta}{5\sec\theta}5\sec^2\theta\,d\theta$$
$$= 25\int_0^{\pi/4}\tan^2\theta\sec\theta\,d\theta.$$

We encountered this indefinite integral in Example 6.4.3 where we found

$$\int \tan^2\theta\sec\theta\,d\theta = \frac{1}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big).$$

So

$$25\int_0^{\pi/4}\tan^2\theta\sec\theta\,d\theta = \frac{25}{2}\big(\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\big)\Big|_0^{\pi/4}$$
$$= \frac{25}{2}\big(\sqrt{2} - \ln(\sqrt{2}+1)\big)$$
$$\approx 6.661.$$

The following equalities are very useful when evaluating integrals using Trigonometric Substitution.

Key Idea 6.4.2 Useful Equalities with Trigonometric Substitution

1. $\sin(2\theta) = 2\sin\theta\cos\theta$
2. $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$
3. $\displaystyle\int \sec^3\theta\,d\theta = \frac{1}{2}\Big(\sec\theta\tan\theta + \ln\big|\sec\theta+\tan\theta\big|\Big) + C$
4. $\displaystyle\int \cos^2\theta\,d\theta = \int \frac{1}{2}\big(1+\cos(2\theta)\big)\,d\theta = \frac{1}{2}\big(\theta + \sin\theta\cos\theta\big) + C.$

The next section introduces Partial Fraction Decomposition, which is an algebraic technique that turns "complicated" fractions into sums of "simpler" fractions, making integration easier.

Exercises 6.4

Terms and Concepts

1. Trigonometric Substitution works on the same principles as Integration by Substitution, though it can feel "________".

2. If one uses Trigonometric Substitution on an integrand containing $\sqrt{25-x^2}$, then one should set $x =$________.

3. Consider the Pythagorean Identity $\sin^2\theta+\cos^2\theta=1$.

   (a) What identity is obtained when both sides are divided by $\cos^2\theta$?

   (b) Use the new identity to simplify $9\tan^2\theta+9$.

4. Why does Key Idea 6.4.1(a) state that $\sqrt{a^2-x^2}=a\cos\theta$, and not $|a\cos\theta|$?

Problems

In Exercises 5 – 16, apply Trigonometric Substitution to evaluate the indefinite integrals.

5. $\displaystyle\int \sqrt{x^2+1}\,dx$

6. $\displaystyle\int \sqrt{x^2+4}\,dx$

7. $\displaystyle\int \sqrt{1-x^2}\,dx$

8. $\displaystyle\int \sqrt{9-x^2}\,dx$

9. $\displaystyle\int \sqrt{x^2-1}\,dx$

10. $\displaystyle\int \sqrt{x^2-16}\,dx$

11. $\displaystyle\int \sqrt{4x^2+1}\,dx$

12. $\displaystyle\int \sqrt{1-9x^2}\,dx$

13. $\displaystyle\int \sqrt{16x^2-1}\,dx$

14. $\displaystyle\int \frac{8}{\sqrt{x^2+2}}\,dx$

15. $\displaystyle\int \frac{3}{\sqrt{7-x^2}}\,dx$

16. $\displaystyle\int \frac{5}{\sqrt{x^2-8}}\,dx$

In Exercises 17 – 26, evaluate the indefinite integrals. Some may be evaluated without Trigonometric Substitution.

17. $\displaystyle\int \frac{\sqrt{x^2-11}}{x}\,dx$

18. $\displaystyle\int \frac{1}{(x^2+1)^2}\,dx$

19. $\displaystyle\int \frac{x}{\sqrt{x^2-3}}\,dx$

20. $\displaystyle\int x^2\sqrt{1-x^2}\,dx$

21. $\displaystyle\int \frac{x}{(x^2+9)^{3/2}}\,dx$

22. $\displaystyle\int \frac{5x^2}{\sqrt{x^2-10}}\,dx$

23. $\displaystyle\int \frac{1}{(x^2+4x+13)^2}\,dx$

24. $\displaystyle\int x^2(1-x^2)^{-3/2}\,dx$

25. $\displaystyle\int \frac{\sqrt{5-x^2}}{7x^2}\,dx$

26. $\displaystyle\int \frac{x^2}{\sqrt{x^2+3}}\,dx$

In Exercises 27 – 32, evaluate the definite integrals by making the proper trigonometric substitution *and* changing the bounds of integration. (Note: each of the corresponding indefinite integrals has appeared previously in this Exercise set.)

27. $\displaystyle\int_{-1}^{1} \sqrt{1-x^2}\,dx$

28. $\displaystyle\int_{4}^{8} \sqrt{x^2-16}\,dx$

29. $\displaystyle\int_{0}^{2} \sqrt{x^2+4}\,dx$

30. $\displaystyle\int_{-1}^{1} \frac{1}{(x^2+1)^2}\,dx$

31. $\displaystyle\int_{-1}^{1} \sqrt{9-x^2}\,dx$

32. $\displaystyle\int_{-1}^{1} x^2\sqrt{1-x^2}\,dx$

6.5 Partial Fraction Decomposition

In this section we investigate the antiderivatives of rational functions. Recall that rational functions are functions of the form $f(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x) \neq 0$. Such functions arise in many contexts, one of which is the solving of certain fundamental differential equations.

We begin with an example that demonstrates the motivation behind this section. Consider the integral $\displaystyle\int \frac{1}{x^2-1}\,dx$. We do not have a simple formula for this (if the denominator were x^2+1, we would recognize the antiderivative as being the arctangent function). It can be solved using Trigonometric Substitution, but note how the integral is easy to evaluate once we realize:

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Thus

$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= \int \frac{1/2}{x-1}\,dx - \int \frac{1/2}{x+1}\,dx\\ &= \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

This section teaches how to *decompose*

$$\frac{1}{x^2-1} \quad \text{into} \quad \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

We start with a rational function $f(x) = \dfrac{p(x)}{q(x)}$, where p and q do not have any common factors and the degree of p is less than the degree of q. Note that in the case of a function $f(x) = \dfrac{p(x)}{q(x)}$ where the degree of p is greater than or equal to that of q, we can perform polynomial long division to rewrite $f(x)$ in the form

$$f(x) = Q(x) + \frac{R(x)}{q(x)},$$

where $Q(x)$ is the quotient polynomial, and $R(x)$, the remainder, always has degree less than that of q.

It can be shown that any polynomial, and hence q, can be factored into a product of linear and irreducible quadratic terms. The following Key Idea states how to decompose a rational function into a sum of rational functions whose denominators are all of lower degree than q.

Key Idea 6.5.1 Partial Fraction Decomposition

Let $\dfrac{p(x)}{q(x)}$ be a rational function, where the degree of p is less than the degree of q.

1. **Linear Terms:** Let $(x-a)$ divide $q(x)$, where $(x-a)^n$ is the highest power of $(x-a)$ that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\cdots+\frac{A_n}{(x-a)^n}.$$

2. **Quadratic Terms:** Let x^2+bx+c divide $q(x)$, where $(x^2+bx+c)^n$ is the highest power of x^2+bx+c that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{B_1x+C_1}{x^2+bx+c}+\frac{B_2x+C_2}{(x^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(x^2+bx+c)^n}.$$

To find the coefficients A_i, B_i and C_i:

1. Multiply all fractions by $q(x)$, clearing the denominators. Collect like terms.
2. Equate the resulting coefficients of the powers of x and solve the resulting system of linear equations.

The following examples will demonstrate how to put this Key Idea into practice. Example 6.5.1 stresses the decomposition aspect of the Key Idea.

Example 6.5.1 Decomposing into partial fractions

Decompose $f(x)=\dfrac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2}$ without solving for the resulting coefficients.

Solution The denominator is already factored, as both x^2+x+2 and x^2+x+7 cannot be factored further. We need to decompose $f(x)$ properly. Since $(x+5)$ is a linear term that divides the denominator, there will be a

$$\frac{A}{x+5}$$

term in the decomposition.

As $(x-2)^3$ divides the denominator, we will have the following terms in the decomposition:

$$\frac{B}{x-2},\quad \frac{C}{(x-2)^2}\quad\text{and}\quad\frac{D}{(x-2)^3}.$$

The x^2+x+2 term in the denominator results in a $\dfrac{Ex+F}{x^2+x+2}$ term.

Finally, the $(x^2+x+7)^2$ term results in the terms

$$\frac{Gx+H}{x^2+x+7}\quad\text{and}\quad\frac{Ix+J}{(x^2+x+7)^2}.$$

All together, we have

$$\frac{1}{(x+5)(x-2)^3(x^2+x+2)(x^2+x+7)^2} = \frac{A}{x+5} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3} + \frac{Ex+F}{x^2+x+2} + \frac{Gx+H}{x^2+x+7} + \frac{Ix+J}{(x^2+x+7)^2}$$

Solving for the coefficients $A, B \ldots J$ would be a bit tedious but not "hard."

Example 6.5.2 Decomposing into partial fractions

Perform the partial fraction decomposition of $\dfrac{1}{x^2-1}$.

Solution The denominator factors into two linear terms: $x^2 - 1 = (x-1)(x+1)$. Thus

$$\frac{1}{x^2-1} = \frac{A}{x-1} + \frac{B}{x+1}.$$

To solve for A and B, first multiply through by $x^2-1 = (x-1)(x+1)$:

$$\begin{aligned} 1 &= \frac{A(x-1)(x+1)}{x-1} + \frac{B(x-1)(x+1)}{x+1} \\ &= A(x+1) + B(x-1) \\ &= Ax + A + Bx - B \end{aligned}$$

Now collect like terms.

$$= (A+B)x + (A-B).$$

The next step is key. Note the equality we have:

$$1 = (A+B)x + (A-B).$$

For clarity's sake, rewrite the left hand side as

$$0x + 1 = (A+B)x + (A-B).$$

On the left, the coefficient of the x term is 0; on the right, it is $(A+B)$. Since both sides are equal, we must have that $0 = A + B$.

Likewise, on the left, we have a constant term of 1; on the right, the constant term is $(A-B)$. Therefore we have $1 = A - B$.

We have two linear equations with two unknowns. This one is easy to solve by hand, leading to

$$\begin{array}{c} A+B=0 \\ A-B=1 \end{array} \quad \Rightarrow \quad \begin{array}{c} A=1/2 \\ B=-1/2 \end{array}.$$

Thus

$$\frac{1}{x^2-1} = \frac{1/2}{x-1} - \frac{1/2}{x+1}.$$

Example 6.5.3 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{1}{(x-1)(x+2)^2}\,dx$.

Solution We decompose the integrand as follows, as described by Key Idea 6.5.1:

$$\frac{1}{(x-1)(x+2)^2} = \frac{A}{x-1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}.$$

To solve for A, B and C, we multiply both sides by $(x-1)(x+2)^2$ and collect like terms:

$$\begin{aligned} 1 &= A(x+2)^2 + B(x-1)(x+2) + C(x-1) \qquad (6.3)\\ &= Ax^2 + 4Ax + 4A + Bx^2 + Bx - 2B + Cx - C \\ &= (A+B)x^2 + (4A+B+C)x + (4A-2B-C) \end{aligned}$$

Note: Equation 6.3 offers a direct route to finding the values of A, B and C. Since the equation holds for all values of x, it holds in particular when $x = 1$. However, when $x = 1$, the right hand side simplifies to $A(1+2)^2 = 9A$. Since the left hand side is still 1, we have $1 = 9A$. Hence $A = 1/9$. Likewise, the equality holds when $x = -2$; this leads to the equation $1 = -3C$. Thus $C = -1/3$.
Knowing A and C, we can find the value of B by choosing yet another value of x, such as $x = 0$, and solving for B.

We have

$$0x^2 + 0x + 1 = (A+B)x^2 + (4A+B+C)x + (4A-2B-C)$$

leading to the equations

$$A + B = 0, \quad 4A + B + C = 0 \quad \text{and} \quad 4A - 2B - C = 1.$$

These three equations of three unknowns lead to a unique solution:

$$A = 1/9, \quad B = -1/9 \quad \text{and} \quad C = -1/3.$$

Thus

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \int \frac{1/9}{x-1}\,dx + \int \frac{-1/9}{x+2}\,dx + \int \frac{-1/3}{(x+2)^2}\,dx.$$

Each can be integrated with a simple substitution with $u = x-1$ or $u = x+2$ (or by directly applying Key Idea 6.1.1 as the denominators are linear functions). The end result is

$$\int \frac{1}{(x-1)(x+2)^2}\,dx = \frac{1}{9}\ln|x-1| - \frac{1}{9}\ln|x+2| + \frac{1}{3(x+2)} + C.$$

Example 6.5.4 Integrating using partial fractions

Use partial fraction decomposition to integrate $\displaystyle\int \frac{x^3}{(x-5)(x+3)}\,dx$.

Note: The values of A and B can be quickly found using the technique described in the margin of Example 6.5.3.

Solution Key Idea 6.5.1 presumes that the degree of the numerator is less than the degree of the denominator. Since this is not the case here, we begin by using polynomial division to reduce the degree of the numerator. We omit the steps, but encourage the reader to verify that

$$\frac{x^3}{(x-5)(x+3)} = x + 2 + \frac{19x+30}{(x-5)(x+3)}.$$

Using Key Idea 6.5.1, we can rewrite the new rational function as:

$$\frac{19x+30}{(x-5)(x+3)} = \frac{A}{x-5} + \frac{B}{x+3}$$

for appropriate values of A and B. Clearing denominators, we have

$$\begin{aligned} 19x + 30 &= A(x+3) + B(x-5) \\ &= (A+B)x + (3A-5B). \end{aligned}$$

This implies that:

$$\begin{aligned} 19 &= A + B \\ 30 &= 3A - 5B. \end{aligned}$$

Solving this system of linear equations gives

$$\begin{aligned} 125/8 &= A \\ 27/8 &= B. \end{aligned}$$

We can now integrate.

$$\begin{aligned} \int \frac{x^3}{(x-5)(x+3)}\,dx &= \int \left(x+2+\frac{125/8}{x-5}+\frac{27/8}{x+3}\right)\,dx \\ &= \frac{x^2}{2}+2x+\frac{125}{8}\ln|x-5|+\frac{27}{8}\ln|x+3|+C. \end{aligned}$$

Example 6.5.5 Integrating using partial fractions

Use partial fraction decomposition to evaluate $\displaystyle\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx.$

Solution The degree of the numerator is less than the degree of the denominator so we begin by applying Key Idea 6.5.1. We have:

$$\frac{7x^2+31x+54}{(x+1)(x^2+6x+11)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+6x+11}.$$

Now clear the denominators.

$$\begin{aligned} 7x^2+31x+54 &= A(x^2+6x+11)+(Bx+C)(x+1) \\ &= (A+B)x^2+(6A+B+C)x+(11A+C). \end{aligned}$$

This implies that:

$$\begin{aligned} 7 &= A+B \\ 31 &= 6A+B+C \\ 54 &= 11A+C. \end{aligned}$$

Solving this system of linear equations gives the nice result of $A=5$, $B=2$ and $C=-1$. Thus

$$\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx = \int\left(\frac{5}{x+1}+\frac{2x-1}{x^2+6x+11}\right)\,dx.$$

The first term of this new integrand is easy to evaluate; it leads to a $5\ln|x+1|$ term. The second term is not hard, but takes several steps and uses substitution techniques.

The integrand $\dfrac{2x-1}{x^2+6x+11}$ has a quadratic in the denominator and a linear term in the numerator. This leads us to try substitution. Let $u=x^2+6x+11$, so $du=(2x+6)\,dx$. The numerator is $2x-1$, not $2x+6$, but we can get a $2x+6$ term in the numerator by adding 0 in the form of "$7-7$."

$$\begin{aligned} \frac{2x-1}{x^2+6x+11} &= \frac{2x-1+7-7}{x^2+6x+11} \\ &= \frac{2x+6}{x^2+6x+11}-\frac{7}{x^2+6x+11}. \end{aligned}$$

We can now integrate the first term with substitution, leading to a $\ln|x^2+6x+11|$ term. The final term can be integrated using arctangent. First, complete the square in the denominator:

$$\frac{7}{x^2+6x+11} = \frac{7}{(x+3)^2+2}.$$

An antiderivative of the latter term can be found using Theorem 6.1.3 and substitution:

$$\int \frac{7}{x^2+6x+11}\,dx = \frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C.$$

Let's start at the beginning and put all of the steps together.

$$\begin{aligned}\int \frac{7x^2+31x+54}{(x+1)(x^2+6x+11)}\,dx &= \int\left(\frac{5}{x+1}+\frac{2x-1}{x^2+6x+11}\right)dx\\ &= \int\frac{5}{x+1}\,dx+\int\frac{2x+6}{x^2+6x+11}\,dx-\int\frac{7}{x^2+6x+11}\,dx\\ &= 5\ln|x+1|+\ln|x^2+6x+11|-\frac{7}{\sqrt{2}}\tan^{-1}\left(\frac{x+3}{\sqrt{2}}\right)+C.\end{aligned}$$

As with many other problems in calculus, it is important to remember that one is not expected to "see" the final answer immediately after seeing the problem. Rather, given the initial problem, we break it down into smaller problems that are easier to solve. The final answer is a combination of the answers of the smaller problems.

Partial Fraction Decomposition is an important tool when dealing with rational functions. Note that at its heart, it is a technique of algebra, not calculus, as we are rewriting a fraction in a new form. Regardless, it is very useful in the realm of calculus as it lets us evaluate a certain set of "complicated" integrals.

The next section introduces new functions, called the Hyperbolic Functions. They will allow us to make substitutions similar to those found when studying Trigonometric Substitution, allowing us to approach even more integration problems.

Exercises 6.5

Terms and Concepts

1. Fill in the blank: Partial Fraction Decomposition is a method of rewriting ________ functions.

2. T/F: It is sometimes necessary to use polynomial division before using Partial Fraction Decomposition.

3. Decompose $\dfrac{1}{x^2-3x}$ without solving for the coefficients, as done in Example 6.5.1.

4. Decompose $\dfrac{7-x}{x^2-9}$ without solving for the coefficients, as done in Example 6.5.1.

5. Decompose $\dfrac{x-3}{x^2-7}$ without solving for the coefficients, as done in Example 6.5.1.

6. Decompose $\dfrac{2x+5}{x^3+7x}$ without solving for the coefficients, as done in Example 6.5.1.

Problems

In Exercises 7 – 26, evaluate the indefinite integral.

7. $\displaystyle\int \frac{7x+7}{x^2+3x-10}\,dx$

8. $\displaystyle\int \frac{7x-2}{x^2+x}\,dx$

9. $\displaystyle\int \frac{-4}{3x^2-12}\,dx$

10. $\displaystyle\int \frac{6x+4}{3x^2+4x+1}\,dx$

11. $\displaystyle\int \frac{x+7}{(x+5)^2}\,dx$

12. $\displaystyle\int \frac{-3x-20}{(x+8)^2}\,dx$

13. $\displaystyle\int \frac{9x^2+11x+7}{x(x+1)^2}\,dx$

14. $\displaystyle\int \frac{-12x^2-x+33}{(x-1)(x+3)(3-2x)}\,dx$

15. $\displaystyle\int \frac{94x^2-10x}{(7x+3)(5x-1)(3x-1)}\,dx$

16. $\displaystyle\int \frac{x^2+x+1}{x^2+x-2}\,dx$

17. $\displaystyle\int \frac{x^3}{x^2-x-20}\,dx$

18. $\displaystyle\int \frac{2x^2-4x+6}{x^2-2x+3}\,dx$

19. $\displaystyle\int \frac{1}{x^3+2x^2+3x}\,dx$

20. $\displaystyle\int \frac{x^2+x+5}{x^2+4x+10}\,dx$

21. $\displaystyle\int \frac{12x^2+21x+3}{(x+1)(3x^2+5x-1)}\,dx$

22. $\displaystyle\int \frac{6x^2+8x-4}{(x-3)(x^2+6x+10)}\,dx$

23. $\displaystyle\int \frac{2x^2+x+1}{(x+1)(x^2+9)}\,dx$

24. $\displaystyle\int \frac{x^2-20x-69}{(x-7)(x^2+2x+17)}\,dx$

25. $\displaystyle\int \frac{9x^2-60x+33}{(x-9)(x^2-2x+11)}\,dx$

26. $\displaystyle\int \frac{6x^2+45x+121}{(x+2)(x^2+10x+27)}\,dx$

In Exercises 27 – 30, evaluate the definite integral.

27. $\displaystyle\int_1^2 \frac{8x+21}{(x+2)(x+3)}\,dx$

28. $\displaystyle\int_0^5 \frac{14x+6}{(3x+2)(x+4)}\,dx$

29. $\displaystyle\int_{-1}^1 \frac{x^2+5x-5}{(x-10)(x^2+4x+5)}\,dx$

30. $\displaystyle\int_0^1 \frac{x}{(x+1)(x^2+2x+1)}\,dx$

6.6 Hyperbolic Functions

The **hyperbolic functions** are a set of functions that have many applications to mathematics, physics, and engineering. Among many other applications, they are used to describe the formation of satellite rings around planets, to describe the shape of a rope hanging from two points, and have application to the theory of special relativity. This section defines the hyperbolic functions and describes many of their properties, especially their usefulness to calculus.

These functions are sometimes referred to as the "hyperbolic trigonometric functions" as there are many, many connections between them and the standard trigonometric functions. Figure 6.6.2 demonstrates one such connection. Just as cosine and sine are used to define points on the circle defined by $x^2+y^2=1$, the functions **hyperbolic cosine** and **hyperbolic sine** are used to define points on the hyperbola $x^2-y^2=1$.

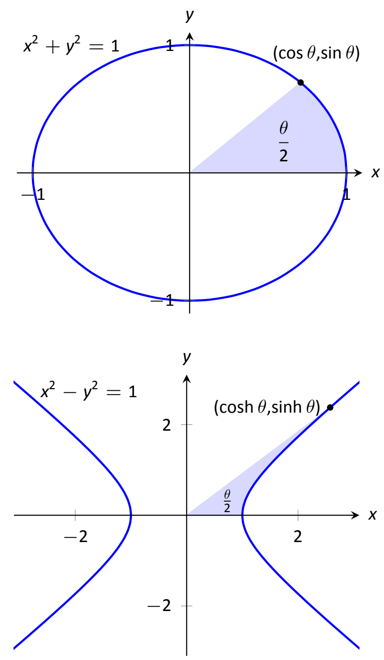


Figure 6.6.2: Using trigonometric functions to define points on a circle and hyperbolic functions to define points on a hyperbola. The area of the shaded regions are included in them.

Definition 6.6.1 Hyperbolic Functions

1. $\cosh x = \dfrac{e^x+e^{-x}}{2}$
2. $\sinh x = \dfrac{e^x-e^{-x}}{2}$
3. $\tanh x = \dfrac{\sinh x}{\cosh x}$
4. $\operatorname{sech} x = \dfrac{1}{\cosh x}$
5. $\operatorname{csch} x = \dfrac{1}{\sinh x}$
6. $\coth x = \dfrac{\cosh x}{\sinh x}$

These hyperbolic functions are graphed in Figure 6.6.1. In the graphs of $\cosh x$ and $\sinh x$, graphs of $e^x/2$ and $e^{-x}/2$ are included with dashed lines. As x gets "large," $\cosh x$ and $\sinh x$ each act like $e^x/2$; when x is a large negative number, $\cosh x$ acts like $e^{-x}/2$ whereas $\sinh x$ acts like $-e^{-x}/2$.

Pronunciation Note:
"cosh" rhymes with "gosh,"
"sinh" rhymes with "pinch," and
"tanh" rhymes with "ranch."

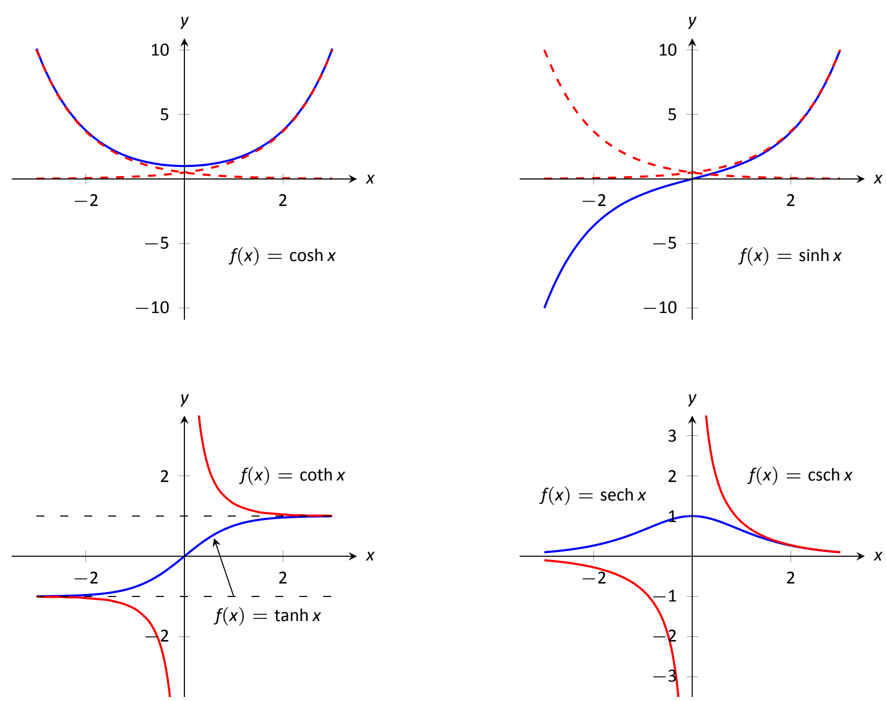


Figure 6.6.1: Graphs of the hyperbolic functions.

Notice the domains of $\tanh x$ and $\operatorname{sech} x$ are $(-\infty, \infty)$, whereas both $\coth x$ and $\operatorname{csch} x$ have vertical asymptotes at $x = 0$. Also note the ranges of these functions, especially $\tanh x$: as $x \to \infty$, both $\sinh x$ and $\cosh x$ approach $e^x/2$, hence $\tanh x$ approaches 1.

The following example explores some of the properties of these functions that bear remarkable resemblance to the properties of their trigonometric counterparts.

Example 6.6.1 Exploring properties of hyperbolic functions

Use Definition 6.6.1 to rewrite the following expressions.

1. $\cosh^2 x - \sinh^2 x$
2. $\tanh^2 x + \operatorname{sech}^2 x$
3. $2\cosh x \sinh x$
4. $\frac{d}{dx}\big(\cosh x\big)$
5. $\frac{d}{dx}\big(\sinh x\big)$
6. $\frac{d}{dx}\big(\tanh x\big)$

Solution

1.
$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2\\ &= \frac{e^{2x} + 2e^xe^{-x} + e^{-2x}}{4} - \frac{e^{2x} - 2e^xe^{-x} + e^{-2x}}{4}\\ &= \frac{4}{4} = 1.\end{aligned}$$

So $\cosh^2 x - \sinh^2 x = 1$.

2.
$$\begin{aligned}\tanh^2 x + \operatorname{sech}^2 x &= \frac{\sinh^2 x}{\cosh^2 x} + \frac{1}{\cosh^2 x}\\ &= \frac{\sinh^2 x + 1}{\cosh^2 x} \qquad \text{Now use identity from \#1.}\\ &= \frac{\cosh^2 x}{\cosh^2 x} = 1.\end{aligned}$$

So $\tanh^2 x + \operatorname{sech}^2 x = 1$.

3.
$$\begin{aligned}2\cosh x \sinh x &= 2\left(\frac{e^x + e^{-x}}{2}\right)\left(\frac{e^x - e^{-x}}{2}\right)\\ &= 2\cdot\frac{e^{2x} - e^{-2x}}{4}\\ &= \frac{e^{2x} - e^{-2x}}{2} = \sinh(2x).\end{aligned}$$

Thus $2\cosh x \sinh x = \sinh(2x)$.

4.
$$\begin{aligned}\frac{d}{dx}\big(\cosh x\big) &= \frac{d}{dx}\left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{e^x - e^{-x}}{2}\\ &= \sinh x.\end{aligned}$$

So $\frac{d}{dx}\big(\cosh x\big) = \sinh x$.

5.
$$\frac{d}{dx}\big(\sinh x\big) = \frac{d}{dx}\left(\frac{e^x - e^{-x}}{2}\right) = \frac{e^x + e^{-x}}{2} = \cosh x.$$

So $\frac{d}{dx}\big(\sinh x\big) = \cosh x$.

6.
$$\frac{d}{dx}\big(\tanh x\big) = \frac{d}{dx}\left(\frac{\sinh x}{\cosh x}\right) = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x.$$

So $\frac{d}{dx}\big(\tanh x\big) = \operatorname{sech}^2 x$.

The following Key Idea summarizes many of the important identities relating to hyperbolic functions. Each can be verified by referring back to Definition 6.6.1.

Key Idea 6.6.1 Useful Hyperbolic Function Properties

Basic Identities

1. $\cosh^2 x - \sinh^2 x = 1$
2. $\tanh^2 x + \operatorname{sech}^2 x = 1$
3. $\coth^2 x - \operatorname{csch}^2 x = 1$
4. $\cosh 2x = \cosh^2 x + \sinh^2 x$
5. $\sinh 2x = 2 \sinh x \cosh x$
6. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$
7. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

Derivatives

1. $\frac{d}{dx}\big(\cosh x\big) = \sinh x$
2. $\frac{d}{dx}\big(\sinh x\big) = \cosh x$
3. $\frac{d}{dx}\big(\tanh x\big) = \operatorname{sech}^2 x$
4. $\frac{d}{dx}\big(\operatorname{sech} x\big) = -\operatorname{sech} x \tanh x$
5. $\frac{d}{dx}\big(\operatorname{csch} x\big) = -\operatorname{csch} x \coth x$
6. $\frac{d}{dx}\big(\coth x\big) = -\operatorname{csch}^2 x$

Integrals

1. $\displaystyle\int \cosh x\, dx = \sinh x + C$
2. $\displaystyle\int \sinh x\, dx = \cosh x + C$
3. $\displaystyle\int \tanh x\, dx = \ln(\cosh x) + C$
4. $\displaystyle\int \coth x\, dx = \ln|\sinh x| + C$

We practice using Key Idea 6.6.1.

Example 6.6.2 Derivatives and integrals of hyperbolic functions

Evaluate the following derivatives and integrals.

1. $\dfrac{d}{dx}\big(\cosh 2x\big)$
2. $\displaystyle\int \operatorname{sech}^2(7t - 3)\, dt$
3. $\displaystyle\int_0^{\ln 2} \cosh x\, dx$

Solution

1. Using the Chain Rule directly, we have $\frac{d}{dx}\big(\cosh 2x\big) = 2 \sinh 2x$.

Just to demonstrate that it works, let's also use the Basic Identity found in Key Idea 6.6.1: $\cosh 2x = \cosh^2 x + \sinh^2 x$.

$$\begin{aligned} \frac{d}{dx}\big(\cosh 2x\big) = \frac{d}{dx}\big(\cosh^2 x + \sinh^2 x\big) &= 2\cosh x \sinh x + 2\sinh x \cosh x \\ &= 4\cosh x \sinh x. \end{aligned}$$

Using another Basic Identity, we can see that $4\cosh x \sinh x = 2\sinh 2x$. We get the same answer either way.

2. We employ substitution, with $u = 7t - 3$ and $du = 7dt$. Applying Key Ideas 6.1.1 and 6.6.1 we have:

$$\int \operatorname{sech}^2(7t-3)\,dt = \frac{1}{7}\tanh(7t-3) + C.$$

3.

$$\int_0^{\ln 2} \cosh x\,dx = \sinh x\Big|_0^{\ln 2} = \sinh(\ln 2) - \sinh 0 = \sinh(\ln 2).$$

We can simplify this last expression as $\sinh x$ is based on exponentials:

$$\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2 - 1/2}{2} = \frac{3}{4}.$$

Inverse Hyperbolic Functions

Just as the inverse trigonometric functions are useful in certain integration problems, the inverse hyperbolic functions are useful with others. Figure 6.6.3 shows the restrictions on the domains to make each function one-to-one and the resulting domains and ranges of their inverse functions. Their graphs are shown in Figure 6.6.4.

Because the hyperbolic functions are defined in terms of exponential functions, their inverses can be expressed in terms of logarithms as shown in Key Idea 6.6.2. It is often more convenient to refer to $\sinh^{-1} x$ than to $\ln\left(x+\sqrt{x^2+1}\right)$, especially when one is working on theory and does not need to compute actual values. On the other hand, when computations are needed, technology is often helpful but many hand-held calculators lack a *convenient* $\sinh^{-1} x$ button. (Often it can be accessed under a menu system, but not conveniently.) In such a situation, the logarithmic representation is useful. The reader is not encouraged to memorize these, but rather know they exist and know how to use them when needed.

Function	Domain	Range
$\cosh x$	$[0,\infty)$	$[1,\infty)$
$\sinh x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\tanh x$	$(-\infty,\infty)$	$(-1,1)$
$\text{sech}\, x$	$[0,\infty)$	$(0,1]$
$\text{csch}\, x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\coth x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,-1)\cup(1,\infty)$

Function	Domain	Range
$\cosh^{-1} x$	$[1,\infty)$	$[0,\infty)$
$\sinh^{-1} x$	$(-\infty,\infty)$	$(-\infty,\infty)$
$\tanh^{-1} x$	$(-1,1)$	$(-\infty,\infty)$
$\text{sech}^{-1} x$	$(0,1]$	$[0,\infty)$
$\text{csch}^{-1} x$	$(-\infty,0)\cup(0,\infty)$	$(-\infty,0)\cup(0,\infty)$
$\coth^{-1} x$	$(-\infty,-1)\cup(1,\infty)$	$(-\infty,0)\cup(0,\infty)$

Figure 6.6.3: Domains and ranges of the hyperbolic and inverse hyperbolic functions.

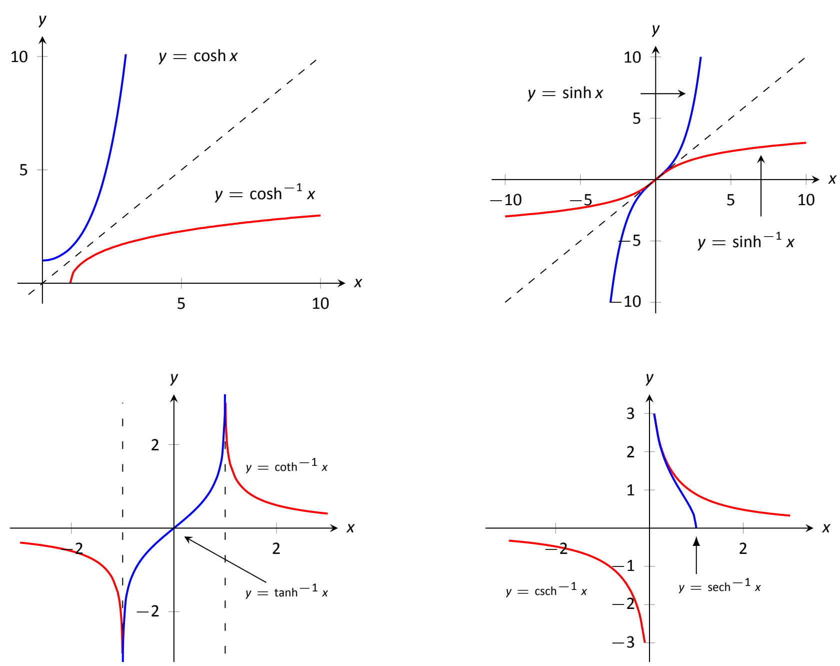


Figure 6.6.4: Graphs of the hyperbolic functions and their inverses.

Key Idea 6.6.2 Logarithmic definitions of Inverse Hyperbolic Functions

1. $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right);\ x \geq 1$
2. $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right);\ |x| < 1$
3. $\operatorname{sech}^{-1} x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right);\ 0 < x \leq 1$
4. $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$
5. $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right);\ |x| > 1$
6. $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right);\ x \neq 0$

The following Key Ideas give the derivatives and integrals relating to the inverse hyperbolic functions. In Key Idea 6.6.4, both the inverse hyperbolic and logarithmic function representations of the antiderivative are given, based on Key Idea 6.6.2. Again, these latter functions are often more useful than the former. Note how inverse hyperbolic functions can be used to solve integrals we used Trigonometric Substitution to solve in Section 6.4.

Key Idea 6.6.3 Derivatives Involving Inverse Hyperbolic Functions

1. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}};\ x > 1$
2. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
3. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2};\ |x| < 1$
4. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}};\ 0 < x < 1$
5. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}};\ x \neq 0$
6. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2};\ |x| > 1$

Key Idea 6.6.4 Integrals Involving Inverse Hyperbolic Functions

1. $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C;\ 0 < a < x \quad = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
2. $\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C;\ a > 0 \quad = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
3. $\int \frac{1}{a^2-x^2}\,dx = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right) + C & x^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{x}{a}\right) + C & a^2 < x^2 \end{cases} \quad = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$
4. $\int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C;\ 0 < x < a \quad = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
5. $\int \frac{1}{x\sqrt{x^2+a^2}}\,dx = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{x}{a}\right| + C;\ x \neq 0,\ a > 0 \quad = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{a^2+x^2}}\right| + C$

We practice using the derivative and integral formulas in the following example.

Example 6.6.3 Derivatives and integrals involving inverse hyperbolic functions

Evaluate the following.

1. $\dfrac{d}{dx}\left[\cosh^{-1}\left(\dfrac{3x-2}{5}\right)\right]$
2. $\displaystyle\int \frac{1}{x^2-1}\,dx$
3. $\displaystyle\int \frac{1}{\sqrt{9x^2+10}}\,dx$

Solution

1. Applying Key Idea 6.6.3 with the Chain Rule gives:
$$\frac{d}{dx}\left[\cosh^{-1}\left(\frac{3x-2}{5}\right)\right] = \frac{1}{\sqrt{\left(\frac{3x-2}{5}\right)^2-1}}\cdot\frac{3}{5}.$$

2. Multiplying the numerator and denominator by (-1) gives: $\displaystyle\int \frac{1}{x^2-1}\,dx = \int \frac{-1}{1-x^2}\,dx$. The second integral can be solved with a direct application of item #3 from Key Idea 6.6.4, with $a=1$. Thus
$$\begin{aligned}\int \frac{1}{x^2-1}\,dx &= -\int \frac{1}{1-x^2}\,dx\\ &= \begin{cases} -\tanh^{-1}(x)+C & x^2<1\\ -\coth^{-1}(x)+C & 1<x^2\end{cases}\\ &= -\frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|+C\\ &= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+C. \qquad (6.4)\end{aligned}$$

We should note that this exact problem is solved at the beginning of Section 6.5. In that example the answer is given as $\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+C$. Note that this is equivalent to the answer given in Equation 6.4, as $\ln(a/b)=\ln a-\ln b$.

3. This requires a substitution, then item #2 of Key Idea 6.6.4 can be applied. Let $u=3x$, hence $du=3dx$. We have
$$\int \frac{1}{\sqrt{9x^2+10}}\,dx = \frac{1}{3}\int \frac{1}{\sqrt{u^2+10}}\,du.$$
Note $a^2=10$, hence $a=\sqrt{10}$. Now apply the integral rule.
$$\begin{aligned}&= \frac{1}{3}\sinh^{-1}\left(\frac{3x}{\sqrt{10}}\right)+C\\ &= \frac{1}{3}\ln\left|3x+\sqrt{9x^2+10}\right|+C.\end{aligned}$$

This section covers a lot of ground. New functions were introduced, along with some of their fundamental identities, their derivatives and antiderivatives, their inverses, and the derivatives and antiderivatives of these inverses. Four Key Ideas were presented, each including quite a bit of information.

Do not view this section as containing a source of information to be memorized, but rather as a reference for future problem solving. Key Idea 6.6.4 contains perhaps the most useful information. Know the integration forms it helps evaluate and understand how to use the inverse hyperbolic answer and the logarithmic answer.

Exercises 6.6

Terms and Concepts

1. In Key Idea 6.6.1, the equation $\int \tanh x\,dx = \ln(\cosh x) + C$ is given. Why is "$\ln|\cosh x|$" not used – i.e., why are absolute values not necessary?

2. The hyperbolic functions are used to define points on the right hand portion of the hyperbola $x^2 - y^2 = 1$, as shown in Figure 6.6.2. How can we use the hyperbolic functions to define points on the left hand portion of the hyperbola?

Problems

In Exercises 3 – 10, verify the given identity using Definition 6.6.1, as done in Example 6.6.1.

3. $\coth^2 x - \operatorname{csch}^2 x = 1$

4. $\cosh 2x = \cosh^2 x + \sinh^2 x$

5. $\cosh^2 x = \dfrac{\cosh 2x + 1}{2}$

6. $\sinh^2 x = \dfrac{\cosh 2x - 1}{2}$

7. $\dfrac{d}{dx}[\operatorname{sech} x] = -\operatorname{sech} x \tanh x$

8. $\dfrac{d}{dx}[\coth x] = -\operatorname{csch}^2 x$

9. $\int \tanh x\,dx = \ln(\cosh x) + C$

10. $\int \coth x\,dx = \ln|\sinh x| + C$

In Exercises 11 – 22, find the derivative of the given function.

11. $f(x) = \sinh 2x$

12. $f(x) = \cosh^2 x$

13. $f(x) = \tanh(x^2)$

14. $f(x) = \ln(\sinh x)$

15. $f(x) = \sinh x \cosh x$

16. $f(x) = x \sinh x - \cosh x$

17. $f(x) = \operatorname{sech}^{-1}(x^2)$

18. $f(x) = \sinh^{-1}(3x)$

19. $f(x) = \cosh^{-1}(2x^2)$

20. $f(x) = \tanh^{-1}(x + 5)$

21. $f(x) = \tanh^{-1}(\cos x)$

22. $f(x) = \cosh^{-1}(\sec x)$

In Exercises 23 – 28, find the equation of the line tangent to the function at the given x-value.

23. $f(x) = \sinh x$ at $x = 0$

24. $f(x) = \cosh x$ at $x = \ln 2$

25. $f(x) = \tanh x$ at $x = -\ln 3$

26. $f(x) = \operatorname{sech}^2 x$ at $x = \ln 3$

27. $f(x) = \sinh^{-1} x$ at $x = 0$

28. $f(x) = \cosh^{-1} x$ at $x = \sqrt{2}$

In Exercises 29 – 44, evaluate the given indefinite integral.

29. $\int \tanh(2x)\,dx$

30. $\int \cosh(3x - 7)\,dx$

31. $\int \sinh x \cosh x\,dx$

32. $\int x \cosh x\,dx$

33. $\int x \sinh x\,dx$

34. $\int \dfrac{1}{\sqrt{x^2 + 1}}\,dx$

35. $\int \dfrac{1}{\sqrt{x^2 - 9}}\,dx$

36. $\int \dfrac{1}{9 - x^2}\,dx$

37. $\int \dfrac{2x}{\sqrt{x^4 - 4}}\,dx$

38. $\int \dfrac{\sqrt{x}}{\sqrt{1 + x^3}}\,dx$

39. $\int \dfrac{1}{x^4 - 16}\,dx$

40. $\int \dfrac{1}{x^2 + x}\,dx$

41. $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$

42. $\displaystyle\int \sinh^{-1} x\,dx$

43. $\displaystyle\int \tanh^{-1} x\,dx$

44. $\displaystyle\int \operatorname{sech} x\,dx$ (Hint: multiply by $\frac{\cosh x}{\cosh x}$; set $u = \sinh x$.)

In Exercises 45 – 48, evaluate the given definite integral.

45. $\displaystyle\int_{-1}^{1} \sinh x\,dx$

46. $\displaystyle\int_{-\ln 2}^{\ln 2} \cosh x\,dx$

47. $\displaystyle\int_{0}^{1} \tanh^{-1} x\,dx$

48. $\displaystyle\int_{0}^{2} \frac{1}{\sqrt{x^2+1}}\,dx$

6.7 L'Hospital's Rule

Our treatment of limits exposed us to the notion of "0/0", an indeterminate form. If

$$\lim_{x\to c} f(x) = 0 \text{ and } \lim_{x\to c} g(x) = 0,$$

we do not conclude that $\lim_{x\to c} f(x)/g(x)$ is $0/0$; rather, we use $0/0$ as notation to describe the fact that both the numerator and denominator approach 0. The expression 0/0 has no numeric value; other work must be done to evaluate the limit.

Other indeterminate forms exist; they are: ∞/∞, $0\cdot\infty$, $\infty-\infty$, 0^0, 1^∞ and ∞^0. Just as "0/0" does not mean "divide 0 by 0," the expression "∞/∞" does not mean "divide infinity by infinity." Instead, it means "a quantity is growing without bound and is being divided by another quantity that is growing without bound." We cannot determine from such a statement what value, if any, results in the limit. Likewise, "$0\cdot\infty$" does not mean "multiply zero by infinity." Instead, it means "one quantity is shrinking to zero, and is being multiplied by a quantity that is growing without bound." We cannot determine from such a description what the result of such a limit will be.

This section introduces l'Hospital's Rule, a method of resolving limits that produce the indeterminate forms 0/0 and ∞/∞. We'll also show how algebraic manipulation can be used to convert other indeterminate expressions into one of these two forms so that our new rule can be applied.

L'Hosptial's Rule is named after Guillaume François Antoine, the Marquis de l'Hosptial, a French mathematician in the late 17th century.
One interesting fact is that L'Hospital's Rule was in fact proved by Johann Bernoulli, whom L'Hospital paid for the right to claim the result as his own in a textbook he produced. (It was not uncommon at the time for members of the nobility to pay to have their name associated with the work of others.) The textbook in question was, in fact, the very first Calculus textbook in recorded history.

Theorem 6.7.1 L'Hospital's Rule, Part 1

Let $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, where *f* and *g* are differentiable functions on an open interval *I* containing *c*, and $g'(x) \neq 0$ on *I* except possibly at *c*. Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}.$$

We demonstrate the use of l'Hospital's Rule in the following examples; we will often use "LHR" as an abbreviation of "l'Hospital's Rule."

Example 6.7.1 Using l'Hospital's Rule

Evaluate the following limits, using l'Hospital's Rule as needed.

1. $\displaystyle\lim_{x\to 0} \frac{\sin x}{x}$
2. $\displaystyle\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x}$
3. $\displaystyle\lim_{x\to 0} \frac{x^2}{1-\cos x}$
4. $\displaystyle\lim_{x\to 2} \frac{x^2+x-6}{x^2-3x+2}$

Solution

1. We proved this limit is 1 in Example 1.3.4 using the Squeeze Theorem. Here we use l'Hospital's Rule to show its power.

$$\lim_{x\to 0} \frac{\sin x}{x} \stackrel{\text{by LHR}}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1.$$

2. $$\lim_{x\to 1} \frac{\sqrt{x+3}-2}{1-x} \stackrel{\text{by LHR}}{=} \lim_{x\to 1} \frac{\frac{1}{2}(x+3)^{-1/2}}{-1} = -\frac{1}{4}.$$

3. $$\lim_{x\to 0}\frac{x^2}{1-\cos x} \stackrel{\text{by LHR}}{=} \lim_{x\to 0}\frac{2x}{\sin x}.$$

This latter limit also evaluates to the 0/0 indeterminate form. To evaluate it, we apply l'Hospital's Rule again.

$$\lim_{x\to 0}\frac{2x}{\sin x} \stackrel{\text{by LHR}}{=} \frac{2}{\cos x} = 2.$$

Thus $\lim_{x\to 0}\frac{x^2}{1-\cos x} = 2.$

4. We already know how to evaluate this limit; first factor the numerator and denominator. We then have:

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2} = \lim_{x\to 2}\frac{(x-2)(x+3)}{(x-2)(x-1)} = \lim_{x\to 2}\frac{x+3}{x-1} = 5.$$

We now show how to solve this using l'Hospital's Rule.

$$\lim_{x\to 2}\frac{x^2+x-6}{x^2-3x+2} \stackrel{\text{by LHR}}{=} \lim_{x\to 2}\frac{2x+1}{2x-3} = 5.$$

Note that at each step where l'Hospital's Rule was applied, it was *needed*: the initial limit returned the indeterminate form of "0/0." If the initial limit returns, for example, 1/2, then l'Hospital's Rule does not apply.

The following theorem extends our initial version of l'Hospital's Rule in two ways. It allows the technique to be applied to the indeterminate form ∞/∞ and to limits where x approaches $\pm\infty$.

Theorem 6.7.2 L'Hospital's Rule, Part 2

1. Let $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, where f and g are differentiable on an open interval I containing a. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

2. Let f and g be differentiable functions on the open interval (a,∞) for some value a, where $g'(x)\neq 0$ on (a,∞) and $\lim_{x\to\infty} f(x)/g(x)$ returns either 0/0 or ∞/∞. Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

A similar statement can be made for limits where x approaches $-\infty$.

Example 6.7.2 Using l'Hospital's Rule with limits involving ∞

Evaluate the following limits.

1. $\lim_{x\to\infty}\frac{3x^2-100x+2}{4x^2+5x-1000}$

2. $\lim_{x\to\infty}\frac{e^x}{x^3}.$

SOLUTION

1. We can evaluate this limit already using Theorem 1.5.1; the answer is 3/4. We apply l'Hospital's Rule to demonstrate its applicability.

$$\lim_{x\to\infty} \frac{3x^2 - 100x + 2}{4x^2 + 5x - 1000} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6x - 100}{8x + 5} \stackrel{\text{by LHR}}{=} \lim_{x\to\infty} \frac{6}{8} = \frac{3}{4}.$$

2. $\lim\limits_{x\to\infty} \frac{e^x}{x^3} \stackrel{\text{by LHR}}{=} \lim\limits_{x\to\infty} \frac{e^x}{3x^2} \stackrel{\text{by LHR}}{=} \lim\limits_{x\to\infty} \frac{e^x}{6x} \stackrel{\text{by LHR}}{=} \lim\limits_{x\to\infty} \frac{e^x}{6} = \infty.$
   Recall that this means that the limit does not exist; as x approaches ∞, the expression e^x/x^3 grows without bound. We can infer from this that e^x grows "faster" than x^3; as x gets large, e^x is far larger than x^3. (This has important implications in computing when considering efficiency of algorithms.)

Indeterminate Forms $0 \cdot \infty$ and $\infty - \infty$

L'Hospital's Rule can only be applied to ratios of functions. When faced with an indeterminate form such as $0\cdot\infty$ or $\infty-\infty$, we can sometimes apply algebra to rewrite the limit so that l'Hospital's Rule can be applied. We demonstrate the general idea in the next example.

Example 6.7.3 Applying l'Hospital's Rule to other indeterminate forms
Evaluate the following limits.

1. $\lim\limits_{x\to 0^+} x \cdot e^{1/x}$
2. $\lim\limits_{x\to 0^-} x \cdot e^{1/x}$
3. $\lim\limits_{x\to\infty} \ln(x+1) - \ln x$
4. $\lim\limits_{x\to\infty} x^2 - e^x$

SOLUTION

1. As $x \to 0^+$, $x \to 0$ and $e^{1/x} \to \infty$. Thus we have the indeterminate form $0 \cdot \infty$. We rewrite the expression $x \cdot e^{1/x}$ as $\dfrac{e^{1/x}}{1/x}$; now, as $x \to 0^+$, we get the indeterminate form ∞/∞ to which l'Hospital's Rule can be applied.

$$\lim_{x\to 0^+} x \cdot e^{1/x} = \lim_{x\to 0^+} \frac{e^{1/x}}{1/x} \stackrel{\text{by LHR}}{=} \lim_{x\to 0^+} \frac{(-1/x^2)e^{1/x}}{-1/x^2} = \lim_{x\to 0^+} e^{1/x} = \infty.$$

   Interpretation: $e^{1/x}$ grows "faster" than x shrinks to zero, meaning their product grows without bound.

2. As $x \to 0^-$, $x \to 0$ and $e^{1/x} \to e^{-\infty} \to 0$. The the limit evaluates to $0 \cdot 0$ which is not an indeterminate form. We conclude then that

$$\lim_{x\to 0^-} x \cdot e^{1/x} = 0.$$

3. This limit initially evaluates to the indeterminate form $\infty-\infty$. By applying a logarithmic rule, we can rewrite the limit as

$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right).$$

   As $x \to \infty$, the argument of the ln term approaches ∞/∞, to which we can apply l'Hospital's Rule.

$$\lim_{x\to\infty} \frac{x+1}{x} \stackrel{\text{by LHR}}{=} \frac{1}{1} = 1.$$

Since $x \to \infty$ implies $\dfrac{x+1}{x} \to 1$, it follows that

$$x \to \infty \quad \text{implies} \quad \ln\left(\frac{x+1}{x}\right) \to \ln 1 = 0.$$

Thus

$$\lim_{x\to\infty} \ln(x+1) - \ln x = \lim_{x\to\infty} \ln\left(\frac{x+1}{x}\right) = 0.$$

Interpretation: since this limit evaluates to 0, it means that for large x, there is essentially no difference between $\ln(x+1)$ and $\ln x$; their difference is essentially 0.

4. The limit $\lim\limits_{x\to\infty} x^2 - e^x$ initially returns the indeterminate form $\infty - \infty$. We can rewrite the expression by factoring out x^2; $x^2 - e^x = x^2\left(1 - \dfrac{e^x}{x^2}\right)$. We need to evaluate how e^x/x^2 behaves as $x \to \infty$:

$$\lim_{x\to\infty} \frac{e^x}{x^2} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \overset{\text{by LHR}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty.$$

Thus $\lim_{x\to\infty} x^2(1 - e^x/x^2)$ evaluates to $\infty \cdot (-\infty)$, which is not an indeterminate form; rather, $\infty \cdot (-\infty)$ evaluates to $-\infty$. We conclude that $\lim\limits_{x\to\infty} x^2 - e^x = -\infty$.

Interpretation: as x gets large, the difference between x^2 and e^x grows very large.

Indeterminate Forms 0^0, 1^∞ and ∞^0

When faced with an indeterminate form that involves a power, it often helps to employ the natural logarithmic function. The following Key Idea expresses the concept, which is followed by an example that demonstrates its use.

Key Idea 6.7.1 Evaluating Limits Involving Indeterminate Forms 0^0, 1^∞ and ∞^0

If $\lim\limits_{x\to c} \ln\big(f(x)\big) = L$, then $\lim\limits_{x\to c} f(x) = \lim\limits_{x\to c} e^{\ln(f(x))} = e^L$.

Example 6.7.4 Using l'Hospital's Rule with indeterminate forms involving exponents

Evaluate the following limits.

1. $\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^x$
2. $\lim\limits_{x\to 0^+} x^x$.

Solution

1. This equivalent to a special limit given in Theorem 1.3.3; these limits have important applications within mathematics and finance. Note that the exponent approaches ∞ while the base approaches 1, leading to the indeterminate form 1^∞. Let $f(x) = (1+1/x)^x$; the problem asks to evaluate

$\lim_{x\to\infty} f(x)$. Let's first evaluate $\lim_{x\to\infty} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to\infty} \ln\big(f(x)\big) &= \lim_{x\to\infty} \ln\left(1+\frac{1}{x}\right)^x \\ &= \lim_{x\to\infty} x\ln\left(1+\frac{1}{x}\right) \\ &= \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x}\right)}{1/x}\end{aligned}$$

This produces the indeterminate form 0/0, so we apply l'Hospital's Rule.

$$\begin{aligned}&= \lim_{x\to\infty} \frac{\frac{1}{1+1/x}\cdot(-1/x^2)}{(-1/x^2)} \\ &= \lim_{x\to\infty} \frac{1}{1+1/x} \\ &= 1.\end{aligned}$$

Thus $\lim_{x\to\infty} \ln\big(f(x)\big) = 1$. We return to the original limit and apply Key Idea 6.7.1.

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = \lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^{\ln(f(x))} = e^1 = e.$$

2. This limit leads to the indeterminate form 0^0. Let $f(x) = x^x$ and consider first $\lim_{x\to0^+} \ln\big(f(x)\big)$.

$$\begin{aligned}\lim_{x\to0^+} \ln\big(f(x)\big) &= \lim_{x\to0^+} \ln\left(x^x\right) \\ &= \lim_{x\to0^+} x\ln x \\ &= \lim_{x\to0^+} \frac{\ln x}{1/x}.\end{aligned}$$

This produces the indeterminate form $-\infty/\infty$ so we apply l'Hospital's Rule.

$$\begin{aligned}&= \lim_{x\to0^+} \frac{1/x}{-1/x^2} \\ &= \lim_{x\to0^+} -x \\ &= 0.\end{aligned}$$

Thus $\lim_{x\to0^+} \ln\big(f(x)\big) = 0$. We return to the original limit and apply Key Idea 6.7.1.

$$\lim_{x\to0^+} x^x = \lim_{x\to0^+} f(x) = \lim_{x\to0^+} e^{\ln(f(x))} = e^0 = 1.$$

This result is supported by the graph of $f(x) = x^x$ given in Figure 6.7.1.

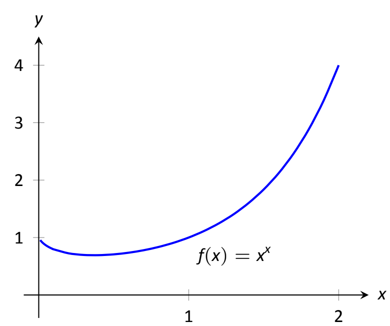


Figure 6.7.1: A graph of $f(x) = x^x$ supporting the fact that as $x \to 0^+, f(x) \to 1$.

Our brief revisit of limits will be rewarded in the next section where we consider *improper integration.* So far, we have only considered definite integrals where the bounds are finite numbers, such as $\int_0^1 f(x)\, dx$. Improper integration considers integrals where one, or both, of the bounds are "infinity." Such integrals have many uses and applications, in addition to generating ideas that are enlightening.

Exercises 6.7

Terms and Concepts

1. List the different indeterminate forms described in this section.

2. T/F: l'Hôpital's Rule provides a faster method of computing derivatives.

3. T/F: l'Hôpital's Rule states that $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)}{g'(x)}$.

4. Explain what the indeterminate form "1^∞" means.

5. Fill in the blanks:

   The Quotient Rule is applied to $\frac{f(x)}{g(x)}$ when taking ________;

   l'Hôpital's Rule is applied to $\frac{f(x)}{g(x)}$ when taking certain ________.

6. Create (but do not evaluate!) a limit that returns "∞^0".

7. Create a function $f(x)$ such that $\lim\limits_{x\to 1} f(x)$ returns "0^0".

8. Create a function $f(x)$ such that $\lim\limits_{x\to\infty} f(x)$ returns "$0 \cdot \infty$".

Problems

In Exercises 9 – 54, evaluate the given limit.

9. $\lim\limits_{x\to 1} \frac{x^2+x-2}{x-1}$

10. $\lim\limits_{x\to 2} \frac{x^2+x-6}{x^2-7x+10}$

11. $\lim\limits_{x\to \pi} \frac{\sin x}{x-\pi}$

12. $\lim\limits_{x\to \pi/4} \frac{\sin x - \cos x}{\cos(2x)}$

13. $\lim\limits_{x\to 0} \frac{\sin(5x)}{x}$

14. $\lim\limits_{x\to 0} \frac{\sin(2x)}{x+2}$

15. $\lim\limits_{x\to 0} \frac{\sin(2x)}{\sin(3x)}$

16. $\lim\limits_{x\to 0} \frac{\sin(ax)}{\sin(bx)}$

17. $\lim\limits_{x\to 0^+} \frac{e^x-1}{x^2}$

18. $\lim\limits_{x\to 0^+} \frac{e^x-x-1}{x^2}$

19. $\lim\limits_{x\to 0^+} \frac{x-\sin x}{x^3-x^2}$

20. $\lim\limits_{x\to \infty} \frac{x^4}{e^x}$

21. $\lim\limits_{x\to \infty} \frac{\sqrt{x}}{e^x}$

22. $\lim\limits_{x\to \infty} \frac{1}{x^2}e^x$

23. $\lim\limits_{x\to \infty} \frac{e^x}{\sqrt{x}}$

24. $\lim\limits_{x\to \infty} \frac{e^x}{2^x}$

25. $\lim\limits_{x\to \infty} \frac{e^x}{3^x}$

26. $\lim\limits_{x\to 3} \frac{x^3-5x^2+3x+9}{x^3-7x^2+15x-9}$

27. $\lim\limits_{x\to -2} \frac{x^3+4x^2+4x}{x^3+7x^2+16x+12}$

28. $\lim\limits_{x\to \infty} \frac{\ln x}{x}$

29. $\lim\limits_{x\to \infty} \frac{\ln(x^2)}{x}$

30. $\lim\limits_{x\to \infty} \frac{\left(\ln x\right)^2}{x}$

31. $\lim\limits_{x\to 0^+} x \cdot \ln x$

32. $\lim\limits_{x\to 0^+} \sqrt{x} \cdot \ln x$

33. $\lim\limits_{x\to 0^+} xe^{1/x}$

34. $\lim\limits_{x\to \infty} x^3 - x^2$

35. $\lim\limits_{x\to \infty} \sqrt{x} - \ln x$

36. $\lim\limits_{x\to -\infty} xe^x$

37. $\lim\limits_{x\to 0^+} \frac{1}{x^2}e^{-1/x}$

38. $\lim\limits_{x\to 0^+} (1+x)^{1/x}$

39. $\lim_{x\to 0^+} (2x)^x$

40. $\lim_{x\to 0^+} (2/x)^x$

41. $\lim_{x\to 0^+} (\sin x)^x$ Hint: use the Squeeze Theorem.

42. $\lim_{x\to 1^+} (1-x)^{1-x}$

43. $\lim_{x\to\infty} (x)^{1/x}$

44. $\lim_{x\to\infty} (1/x)^x$

45. $\lim_{x\to 1^+} (\ln x)^{1-x}$

46. $\lim_{x\to\infty} (1+x)^{1/x}$

47. $\lim_{x\to\infty} (1+x^2)^{1/x}$

48. $\lim_{x\to\pi/2} \tan x \cos x$

49. $\lim_{x\to\pi/2} \tan x \sin(2x)$

50. $\lim_{x\to 1^+} \frac{1}{\ln x} - \frac{1}{x-1}$

51. $\lim_{x\to 3^+} \frac{5}{x^2-9} - \frac{x}{x-3}$

52. $\lim_{x\to\infty} x\tan(1/x)$

53. $\lim_{x\to\infty} \frac{(\ln x)^3}{x}$

54. $\lim_{x\to 1} \frac{x^2+x-2}{\ln x}$

6.8 Improper Integration

We begin this section by considering the following definite integrals:

- $\displaystyle\int_0^{100} \frac{1}{1+x^2}\,dx \approx 1.5608,$

- $\displaystyle\int_0^{1000} \frac{1}{1+x^2}\,dx \approx 1.5698,$

- $\displaystyle\int_0^{10,000} \frac{1}{1+x^2}\,dx \approx 1.5707.$

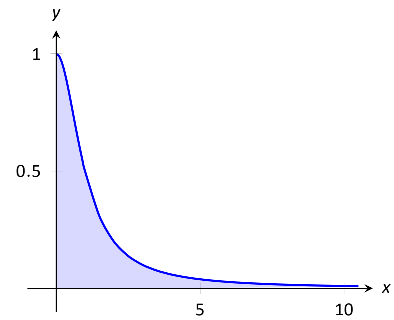


Figure 6.8.1: Graphing $f(x) = \dfrac{1}{1+x^2}$.

Notice how the integrand is $1/(1+x^2)$ in each integral (which is sketched in Figure 6.8.1). As the upper bound gets larger, one would expect the "area under the curve" would also grow. While the definite integrals do increase in value as the upper bound grows, they are not increasing by much. In fact, consider:

$$\int_0^b \frac{1}{1+x^2}\,dx = \tan^{-1} x\Big|_0^b = \tan^{-1} b - \tan^{-1} 0 = \tan^{-1} b.$$

As $b \to \infty$, $\tan^{-1} b \to \pi/2$. Therefore it seems that as the upper bound b grows, the value of the definite integral $\displaystyle\int_0^b \frac{1}{1+x^2}\,dx$ approaches $\pi/2 \approx 1.5708$. This should strike the reader as being a bit amazing: even though the curve extends "to infinity," it has a finite amount of area underneath it.

When we defined the definite integral $\displaystyle\int_a^b f(x)\,dx$, we made two stipulations:

1. The interval over which we integrated, $[a, b]$, was a finite interval, and

2. The function $f(x)$ was continuous on $[a, b]$ (ensuring that the range of f was finite).

In this section we consider integrals where one or both of the above conditions do not hold. Such integrals are called **improper integrals.**

Improper Integrals with Infinite Bounds

Definition 6.8.1 Improper Integrals with Infinite Bounds; Converge, Diverge

1. Let f be a continuous function on $[a,\infty)$. Define

$$\int_a^\infty f(x)\,dx \quad \text{to be} \quad \lim_{b\to\infty}\int_a^b f(x)\,dx.$$

2. Let f be a continuous function on $(-\infty,b]$. Define

$$\int_{-\infty}^b f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^b f(x)\,dx.$$

3. Let f be a continuous function on $(-\infty,\infty)$. Let c be any real number; define

$$\int_{-\infty}^\infty f(x)\,dx \quad \text{to be} \quad \lim_{a\to-\infty}\int_a^c f(x)\,dx + \lim_{b\to\infty}\int_c^b f(x)\,dx.$$

An improper integral is said to **converge** if its corresponding limit exists; otherwise, it **diverges**. The improper integral in part 3 converges if and only if both of its limits exist.

Example 6.8.1 Evaluating improper integrals

Evaluate the following improper integrals.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
3. $\displaystyle\int_{-\infty}^0 e^x\,dx$
4. $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx$

Solution

1.
$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x^2}\,dx = \lim_{b\to\infty}\left.\frac{-1}{x}\right|_1^b$$
$$= \lim_{b\to\infty}\frac{-1}{b}+1$$
$$= 1.$$

A graph of the area defined by this integral is given in Figure 6.8.2.

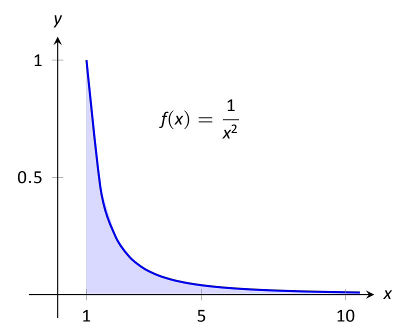


Figure 6.8.2: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.8.1.

2.
$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$
$$= \lim_{b\to\infty} \ln|x|\Big|_1^b$$
$$= \lim_{b\to\infty} \ln(b)$$
$$= \infty.$$

The limit does not exist, hence the improper integral $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ diverges.

Compare the graphs in Figures 6.8.2 and 6.8.3; notice how the graph of $f(x) = 1/x$ is noticeably larger. This difference is enough to cause the improper integral to diverge.

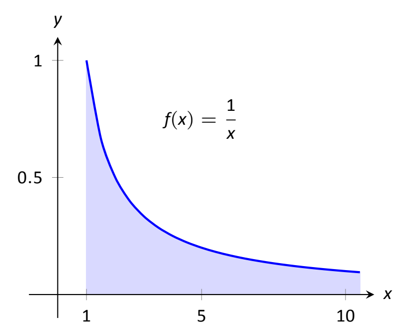


Figure 6.8.3: A graph of $f(x) = \frac{1}{x}$ in Example 6.8.1.

3.
$$\int_{-\infty}^{0} e^x\,dx = \lim_{a\to-\infty}\int_a^0 e^x\,dx$$
$$= \lim_{a\to-\infty} e^x\Big|_a^0$$
$$= \lim_{a\to-\infty} e^0 - e^a$$
$$= 1.$$

A graph of the area defined by this integral is given in Figure 6.8.4.

4. We will need to break this into two improper integrals and choose a value of c as in part 3 of Definition 6.8.1. Any value of c is fine; we choose $c = 0$.

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx$$
$$= \lim_{a\to-\infty} \tan^{-1}x\Big|_a^0 + \lim_{b\to\infty} \tan^{-1}x\Big|_0^b$$
$$= \lim_{a\to-\infty}\left(\tan^{-1}0 - \tan^{-1}a\right) + \lim_{b\to\infty}\left(\tan^{-1}b - \tan^{-1}0\right)$$
$$= \left(0 - \frac{-\pi}{2}\right) + \left(\frac{\pi}{2} - 0\right).$$

Each limit exists, hence the original integral converges and has value:

$$= \pi.$$

A graph of the area defined by this integral is given in Figure 6.8.5.

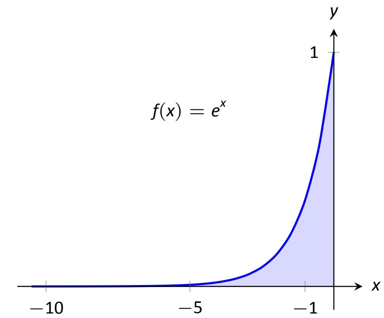


Figure 6.8.4: A graph of $f(x) = e^x$ in Example 6.8.1.

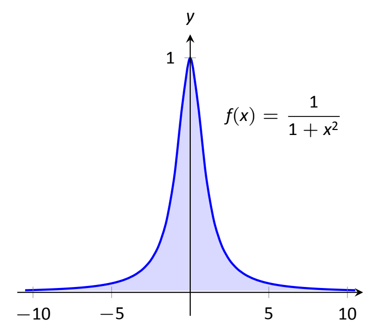


Figure 6.8.5: A graph of $f(x) = \frac{1}{1+x^2}$ in Example 6.8.1.

Recall that many limits that result in indeterminate forms can be handled using **l'Hospital's Rule**. We briefly recall the statement of the theorem: suppose that functions f and g are differentiable on an open interval containing a, and that either

$$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

or

$$\lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = \infty.$$

Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided that the latter limit exists. It is not uncommon for the limits resulting from improper integrals to need this rule as demonstrated next.

Note that l'Hosptial's rule can also be applied in the case of limits where $x \to \pm\infty$.

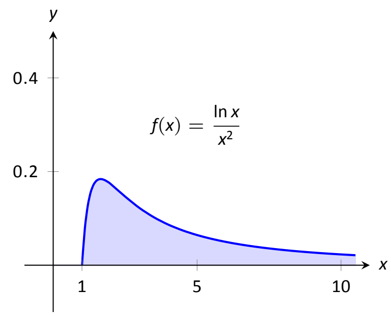


Figure 6.8.6: A graph of $f(x) = \frac{\ln x}{x^2}$ in Example 6.8.2.

Example 6.8.2 Improper integration and l'Hospital's Rule

Evaluate the improper integral $\displaystyle\int_1^\infty \frac{\ln x}{x^2}\,dx$.

Solution This integral will require the use of Integration by Parts. Let $u = \ln x$ and $dv = 1/x^2\,dx$. Then

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx\\ &= \lim_{b\to\infty}\left(-\frac{\ln x}{x}\Big|_1^b + \int_1^b \frac{1}{x^2}\,dx\right)\\ &= \lim_{b\to\infty}\left(-\frac{\ln x}{x} - \frac{1}{x}\right)\Big|_1^b\\ &= \lim_{b\to\infty}\left(-\frac{\ln b}{b} - \frac{1}{b} - (-\ln 1 - 1)\right).\end{aligned}$$

The $1/b$ and $\ln 1$ terms go to 0, leaving $\displaystyle\lim_{b\to\infty} -\frac{\ln b}{b} + 1$. We need to evaluate $\displaystyle\lim_{b\to\infty}\frac{\ln b}{b}$ with l'Hospital's Rule. We have:

$$\begin{aligned}\lim_{b\to\infty}\frac{\ln b}{b} &\overset{\text{by LHR}}{=} \lim_{b\to\infty}\frac{1/b}{1}\\ &= 0.\end{aligned}$$

Thus the improper integral evaluates as:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = 1.$$

Note: In Definition 6.8.2, c can be one of the endpoints (a or b). In that case, there is only one limit to consider as part of the definition.

Improper Integrals with Infinite Range

We have just considered definite integrals where the interval of integration was infinite. We now consider another type of improper integration, where the range of the integrand is infinite.

Definition 6.8.2 Improper Integration with Infinite Range

Let $f(x)$ be a continuous function on $[a, b]$ except at c, $a \le c \le b$, where $x = c$ is a vertical asymptote of f. Define

$$\int_a^b f(x)\,dx = \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx.$$

Example 6.8.3 Improper integration of functions with infinite range

Evaluate the following improper integrals:

1. $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx$ 2. $\displaystyle\int_{-1}^1 \frac{1}{x^2}\,dx.$

SOLUTION

1. A graph of $f(x) = 1/\sqrt{x}$ is given in Figure 6.8.7. Notice that f has a vertical asymptote at $x = 0$; in some sense, we are trying to compute the area of a region that has no "top." Could this have a finite value?

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x}}\,dx &= \lim_{a\to 0^+}\int_a^1 \frac{1}{\sqrt{x}}\,dx\\ &= \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^1\\ &= \lim_{a\to 0^+} 2\left(\sqrt{1}-\sqrt{a}\right)\\ &= 2.\end{aligned}$$

It turns out that the region does have a finite area even though it has no upper bound (strange things can occur in mathematics when considering the infinite).

2. The function $f(x) = 1/x^2$ has a vertical asymptote at $x = 0$, as shown in Figure 6.8.8, so this integral is an improper integral. Let's eschew using limits for a moment and proceed without recognizing the improper nature of the integral. This leads to:

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,dx &= -\frac{1}{x}\Big|_{-1}^1\\ &= -1-(1)\\ &= -2.\ (!)\end{aligned}$$

Clearly the area in question is above the x-axis, yet the area is supposedly negative! Why does our answer not match our intuition? To answer this, evaluate the integral using Definition 6.8.2.

$$\begin{aligned}\int_{-1}^1 \frac{1}{x^2}\,dx &= \lim_{t\to 0^-}\int_{-1}^t \frac{1}{x^2}\,dx + \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx\\ &= \lim_{t\to 0^-} -\frac{1}{x}\Big|_{-1}^t + \lim_{t\to 0^+} -\frac{1}{x}\Big|_t^1\\ &= \lim_{t\to 0^-} -\frac{1}{t} - 1 + \lim_{t\to 0^+} -1 + \frac{1}{t}\\ &\Rightarrow \Big(\infty - 1\Big) + \Big(-1+\infty\Big).\end{aligned}$$

Neither limit converges hence the original improper integral diverges. The nonsensical answer we obtained by ignoring the improper nature of the integral is just that: nonsensical.

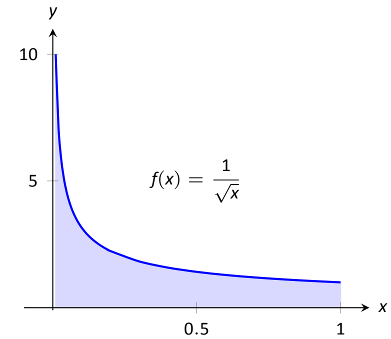


Figure 6.8.7: A graph of $f(x) = \frac{1}{\sqrt{x}}$ in Example 6.8.3.

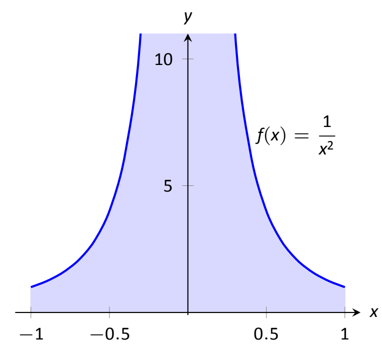


Figure 6.8.8: A graph of $f(x) = \frac{1}{x^2}$ in Example 6.8.3.

Understanding Convergence and Divergence

Oftentimes we are interested in knowing simply whether or not an improper integral converges, and not necessarily the value of a convergent integral. We provide here several tools that help determine the convergence or divergence of improper integrals without integrating.

Our first tool is to understand the behaviour of functions of the form $\dfrac{1}{x^p}$.

Example 6.8.4 Improper integration of $1/x^p$

Determine the values of p for which $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges.

Solution We begin by integrating and then evaluating the limit.

$$\begin{aligned}\int_1^\infty \frac{1}{x^p}\,dx &= \lim_{b\to\infty}\int_1^b \frac{1}{x^p}\,dx\\ &= \lim_{b\to\infty}\int_1^b x^{-p}\,dx \qquad \text{(assume } p\neq 1\text{)}\\ &= \lim_{b\to\infty}\frac{1}{-p+1}x^{-p+1}\Big|_1^b\\ &= \lim_{b\to\infty}\frac{1}{1-p}\left(b^{1-p}-1^{1-p}\right).\end{aligned}$$

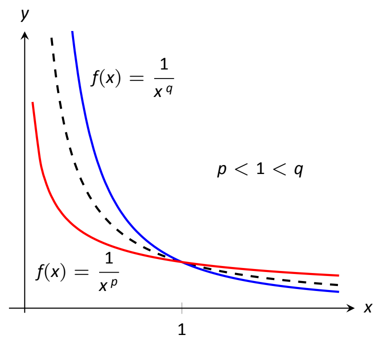


Figure 6.8.9: Plotting functions of the form $1/x^p$ in Example 6.8.4.

When does this limit converge – i.e., when is this limit *not* ∞? This limit converges precisely when the power of b is less than 0: when $1-p<0 \Rightarrow 1<p$.

Our analysis shows that if $p>1$, then $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges. When $p<1$ the improper integral diverges; we showed in Example 6.8.1 that when $p=1$ the integral also diverges.

Figure 6.8.9 graphs $y=1/x$ with a dashed line, along with graphs of $y=1/x^p$, $p<1$, and $y=1/x^q$, $q>1$. Somehow the dashed line forms a dividing line between convergence and divergence.

The result of Example 6.8.4 provides an important tool in determining the convergence of other integrals. A similar result is proved in the exercises about improper integrals of the form $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$. These results are summarized in the following Key Idea.

Key Idea 6.8.1 Convergence of Improper Integrals $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ and $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$.

1. The improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converges when $p>1$ and diverges when $p\leq 1$.
2. The improper integral $\displaystyle\int_0^1 \frac{1}{x^p}\,dx$ converges when $p<1$ and diverges when $p\geq 1$.

A basic technique in determining convergence of improper integrals is to compare an integrand whose convergence is unknown to an integrand whose convergence is known. We often use integrands of the form $1/x^p$ to compare

Note: We used the upper and lower bound of "1" in Key Idea 6.8.1 for convenience. It can be replaced by any a where $a>0$.

to as their convergence on certain intervals is known. This is described in the following theorem.

Theorem 6.8.1 Direct Comparison Test for Improper Integrals

Let f and g be continuous on $[a, \infty)$ where $0 \leq f(x) \leq g(x)$ for all x in $[a, \infty)$.

1. If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.
2. If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.

Example 6.8.5 Determining convergence of improper integrals

Determine the convergence of the following improper integrals.

1. $\int_1^\infty e^{-x^2}\,dx$ 2. $\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$

Solution

1. The function $f(x) = e^{-x^2}$ does not have an antiderivative expressible in terms of elementary functions, so we cannot integrate directly. It is comparable to $g(x) = 1/x^2$, and as demonstrated in Figure 6.8.10, $e^{-x^2} < 1/x^2$ on $[1, \infty)$. We know from Key Idea 6.8.1 that $\int_1^\infty \frac{1}{x^2}\,dx$ converges, hence $\int_1^\infty e^{-x^2}\,dx$ also converges.

2. Note that for large values of x, $\frac{1}{\sqrt{x^2-x}} \approx \frac{1}{\sqrt{x^2}} = \frac{1}{x}$. We know from Key Idea 6.8.1 and the subsequent note that $\int_3^\infty \frac{1}{x}\,dx$ diverges, so we seek to compare the original integrand to $1/x$.

   It is easy to see that when $x > 0$, we have $x = \sqrt{x^2} > \sqrt{x^2-x}$. Taking reciprocals reverses the inequality, giving
   $$\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}.$$
   Using Theorem 6.8.1, we conclude that since $\int_3^\infty \frac{1}{x}\,dx$ diverges, $\int_3^\infty \frac{1}{\sqrt{x^2-x}}\,dx$ diverges as well. Figure 6.8.11 illustrates this.

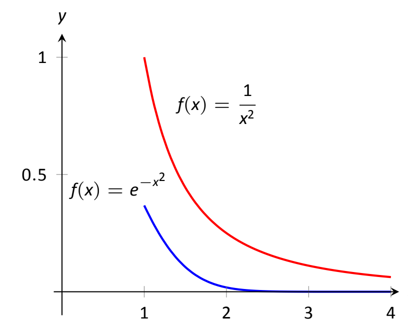


Figure 6.8.10: Graphs of $f(x) = e^{-x^2}$ and $f(x) = 1/x^2$ in Example 6.8.5.

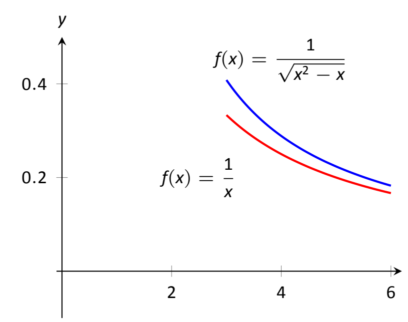


Figure 6.8.11: Graphs of $f(x) = 1/\sqrt{x^2-x}$ and $f(x) = 1/x$ in Example 6.8.5.

Being able to compare "unknown" integrals to "known" integrals is very useful in determining convergence. However, some of our examples were a little "too nice." For instance, it was convenient that $\frac{1}{x} < \frac{1}{\sqrt{x^2-x}}$, but what if the "$-x$" were replaced with a "$+2x+5$"? That is, what can we say about the convergence of $\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$? We have $\frac{1}{x} > \frac{1}{\sqrt{x^2+2x+5}}$, so we cannot use Theorem 6.8.1.

In cases like this (and many more) it is useful to employ the following theorem.

Theorem 6.8.2 Limit Comparison Test for Improper Integrals

Let f and g be continuous functions on $[a, \infty)$ where $f(x) > 0$ and $g(x) > 0$ for all x. If

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge or both diverge.

Example 6.8.6 Determining convergence of improper integrals

Determine the convergence of $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$.

Solution As x gets large, the denominator of the integrand will begin to behave much like $y = x$. So we compare $\dfrac{1}{\sqrt{x^2+2x+5}}$ to $\dfrac{1}{x}$ with the Limit Comparison Test:

$$\lim_{x\to\infty} \frac{1/\sqrt{x^2+2x+5}}{1/x} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+2x+5}}.$$

The immediate evaluation of this limit returns ∞/∞, an indeterminate form. Using l'Hospital's Rule seems appropriate, but in this situation, it does not lead to useful results. (We encourage the reader to employ l'Hospital's Rule at least once to verify this.)

The trouble is the square root function. To get rid of it, we employ the following fact: If $\lim_{x\to c} f(x) = L$, then $\lim_{x\to c} f(x)^2 = L^2$. (This is true when either c or L is ∞.) So we consider now the limit

$$\lim_{x\to\infty} \frac{x^2}{x^2+2x+5}.$$

This converges to 1, meaning the original limit also converged to 1. As x gets very large, the function $\dfrac{1}{\sqrt{x^2+2x+5}}$ looks very much like $\dfrac{1}{x}$. Since we know that $\displaystyle\int_3^\infty \frac{1}{x}\,dx$ diverges, by the Limit Comparison Test we know that $\displaystyle\int_3^\infty \frac{1}{\sqrt{x^2+2x+5}}\,dx$ also diverges. Figure 6.8.12 graphs $f(x) = 1/\sqrt{x^2+2x+5}$ and $f(x) = 1/x$, illustrating that as x gets large, the functions become indistinguishable.

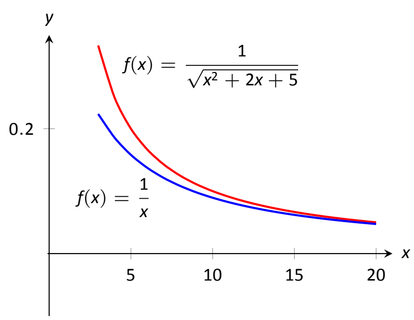


Figure 6.8.12: Graphing $f(x) = \frac{1}{\sqrt{x^2+2x+5}}$ and $f(x) = \frac{1}{x}$ in Example 6.8.6.

Both the Direct and Limit Comparison Tests were given in terms of integrals over an infinite interval. There are versions that apply to improper integrals with an infinite range, but as they are a bit wordy and a little more difficult to employ, they are omitted from this text.

This chapter has explored many integration techniques. We learned Substitution, which "undoes" the Chain Rule of differentiation, as well as Integration by Parts, which "undoes" the Product Rule. We learned specialized techniques for handling trigonometric functions and introduced the hyperbolic functions, which are closely related to the trigonometric functions. All techniques effectively have this goal in common: rewrite the integrand in a new way so that the integration step is easier to see and implement.

As stated before, integration is, in general, hard. It is easy to write a function whose antiderivative is impossible to write in terms of elementary functions, and even when a function does have an antiderivative expressible by elementary functions, it may be really hard to discover what it is. The powerful computer algebra system *Mathematica*® has approximately 1,000 pages of code dedicated to integration.

Do not let this difficulty discourage you. There is great value in learning integration techniques, as they allow one to manipulate an integral in ways that can illuminate a concept for greater understanding. There is also great value in understanding the need for good numerical techniques: the Trapezoidal and Simpson's Rules are just the beginning of powerful techniques for approximating the value of integration.

The next chapter stresses the uses of integration. We generally do not find antiderivatives for antiderivative's sake, but rather because they provide the solution to some type of problem. The following chapter introduces us to a number of different problems whose solution is provided by integration.

Exercises 6.8

Terms and Concepts

1. The definite integral was defined with what two stipulations?

2. If $\lim_{b\to\infty}\int_0^b f(x)\,dx$ exists, then the integral $\int_0^\infty f(x)\,dx$ is said to ________________.

3. If $\int_1^\infty f(x)\,dx = 10$, and $0 \leq g(x) \leq f(x)$ for all x, then we know that $\int_1^\infty g(x)\,dx$ ________________.

4. For what values of p will $\int_1^\infty \frac{1}{x^p}\,dx$ converge?

5. For what values of p will $\int_{10}^\infty \frac{1}{x^p}\,dx$ converge?

6. For what values of p will $\int_0^1 \frac{1}{x^p}\,dx$ converge?

Problems

In Exercises 7 – 34, evaluate the given improper integral.

7. $\int_0^\infty e^{5-2x}\,dx$

8. $\int_1^\infty \frac{1}{x^3}\,dx$

9. $\int_1^\infty x^{-4}\,dx$

10. $\int_{-\infty}^\infty \frac{1}{x^2+9}\,dx$

11. $\int_{-\infty}^0 2^x\,dx$

12. $\int_{-\infty}^0 \left(\frac{1}{2}\right)^x\,dx$

13. $\int_{-\infty}^\infty \frac{x}{x^2+1}\,dx$

14. $\int_3^\infty \frac{1}{x^2-4}\,dx$

15. $\int_2^\infty \frac{1}{(x-1)^2}\,dx$

16. $\int_1^2 \frac{1}{(x-1)^2}\,dx$

17. $\int_2^\infty \frac{1}{x-1}\,dx$

18. $\int_1^2 \frac{1}{x-1}\,dx$

19. $\int_{-1}^1 \frac{1}{x}\,dx$

20. $\int_1^3 \frac{1}{x-2}\,dx$

21. $\int_0^\pi \sec^2 x\,dx$

22. $\int_{-2}^1 \frac{1}{\sqrt{|x|}}\,dx$

23. $\int_0^\infty xe^{-x}\,dx$

24. $\int_0^\infty xe^{-x^2}\,dx$

25. $\int_{-\infty}^\infty xe^{-x^2}\,dx$

26. $\int_{-\infty}^\infty \frac{1}{e^x+e^{-x}}\,dx$

27. $\int_0^1 x\ln x\,dx$

28. $\int_0^1 x^2\ln x\,dx$

29. $\int_1^\infty \frac{\ln x}{x}\,dx$

30. $\int_0^1 \ln x\,dx$

31. $\int_1^\infty \frac{\ln x}{x^2}\,dx$

32. $\int_1^\infty \frac{\ln x}{\sqrt{x}}\,dx$

33. $\int_0^\infty e^{-x}\sin x\,dx$

34. $\int_0^\infty e^{-x}\cos x\,dx$

In Exercises 35 – 44, use the Direct Comparison Test or the Limit Comparison Test to determine whether the given definite integral converges or diverges. Clearly state what test is being used and what function the integrand is being compared to.

35. $\displaystyle\int_{10}^{\infty} \frac{3}{\sqrt{3x^2+2x-5}}\,dx$

36. $\displaystyle\int_{2}^{\infty} \frac{4}{\sqrt{7x^3-x}}\,dx$

37. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x+3}}{\sqrt{x^3-x^2+x+1}}\,dx$

38. $\displaystyle\int_{1}^{\infty} e^{-x}\ln x\,dx$

39. $\displaystyle\int_{5}^{\infty} e^{-x^2+3x+1}\,dx$

40. $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{e^x}\,dx$

41. $\displaystyle\int_{2}^{\infty} \frac{1}{x^2+\sin x}\,dx$

42. $\displaystyle\int_{0}^{\infty} \frac{x}{x^2+\cos x}\,dx$

43. $\displaystyle\int_{0}^{\infty} \frac{1}{x+e^x}\,dx$

44. $\displaystyle\int_{0}^{\infty} \frac{1}{e^x-x}\,dx$

7: Applications of Integration

We begin this chapter with a reminder of a few key concepts from Chapter 5. Let f be a continuous function on $[a, b]$ which is partitioned into n equally spaced subintervals as

$$a < x_1 < x_2 < \cdots < x_n < x_{n+1} = b.$$

Let $\Delta x = (b-a)/n$ denote the length of the subintervals, and let c_i be any x-value in the i^{th} subinterval. Definition 5.3.2 states that the sum

$$\sum_{i=1}^{n} f(c_i)\Delta x$$

is a *Riemann Sum.* Riemann Sums are often used to approximate some quantity (area, volume, work, pressure, etc.). The *approximation* becomes *exact* by taking the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Theorem 5.3.2 connects limits of Riemann Sums to definite integrals:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\, dx.$$

Finally, the Fundamental Theorem of Calculus states how definite integrals can be evaluated using antiderivatives.

This chapter employs the following technique to a variety of applications. Suppose the value Q of a quantity is to be calculated. We first approximate the value of Q using a Riemann Sum, then find the exact value via a definite integral. We spell out this technique in the following Key Idea.

Key Idea 7.0.1 Application of Definite Integrals Strategy

Let a quantity be given whose value Q is to be computed.

1. Divide the quantity into n smaller "subquantities" of value Q_i.
2. Identify a variable x and function $f(x)$ such that each subquantity can be approximated with the product $f(c_i)\Delta x$, where Δx represents a small change in x. Thus $Q_i \approx f(c_i)\Delta x$. A sample approximation $f(c_i)\Delta x$ of Q_i is called a *differential element*.
3. Recognize that $Q = \sum_{i=1}^{n} Q_i \approx \sum_{i=1}^{n} f(c_i)\Delta x$, which is a Riemann Sum.
4. Taking the appropriate limit gives $Q = \int_a^b f(x)\, dx$

This Key Idea will make more sense after we have had a chance to use it several times. We begin with Area Between Curves, which we addressed briefly in Section 5.5.4.

7.1 Area Between Curves

We are often interested in knowing the area of a region. Forget momentarily that we addressed this already in Section 5.5.4 and approach it instead using the technique described in Key Idea 7.0.1.

Let Q be the area of a region bounded by continuous functions f and g. If we break the region into many subregions, we have an obvious equation:

Total Area = sum of the areas of the subregions.

The issue to address next is how to systematically break a region into subregions. A graph will help. Consider Figure 7.1.1 (a) where a region between two curves is shaded. While there are many ways to break this into subregions, one particularly efficient way is to "slice" it vertically, as shown in Figure 7.1.1 (b), into n equally spaced slices.

We now approximate the area of a slice. Again, we have many options, but using a rectangle seems simplest. Picking any x-value c_i in the i^{th} slice, we set the height of the rectangle to be $f(c_i) - g(c_i)$, the difference of the corresponding y-values. The width of the rectangle is a small difference in x-values, which we represent with Δx. Figure 7.1.1 (c) shows sample points c_i chosen in each subinterval and appropriate rectangles drawn. (Each of these rectangles represents a differential element.) Each slice has an area approximately equal to $\big(f(c_i) - g(c_i)\big)\Delta x$; hence, the total area is approximately the Riemann Sum

$$Q = \sum_{i=1}^{n} \big(f(c_i) - g(c_i)\big)\Delta x.$$

Taking the limit as $n \to \infty$ gives the exact area as $\int_a^b \big(f(x) - g(x)\big)\,dx$.

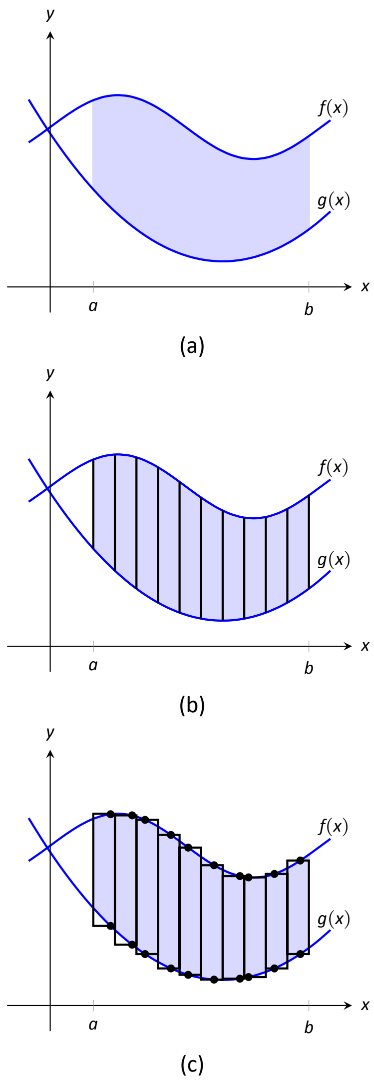


Figure 7.1.1: Subdividing a region into vertical slices and approximating the areas with rectangles.

Theorem 7.1.1 Area Between Curves (restatement of Theorem 5.4.3)

Let $f(x)$ and $g(x)$ be continuous functions defined on $[a, b]$ where $f(x) \geq g(x)$ for all x in $[a, b]$. The area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $x = b$ is

$$\int_a^b \big(f(x) - g(x)\big)\,dx.$$

Example 7.1.1 Finding area enclosed by curves

Find the area of the region bounded by $f(x) = \sin x + 2$, $g(x) = \frac{1}{2}\cos(2x) - 1$, $x = 0$ and $x = 4\pi$, as shown in Figure 7.1.2.

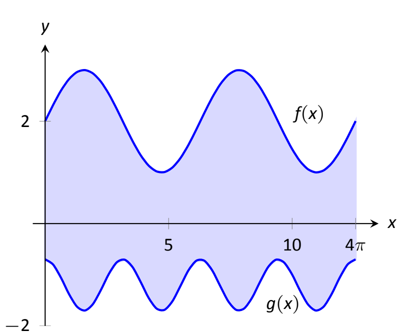


Figure 7.1.2: Graphing an enclosed region in Example 7.1.1.

Solution The graph verifies that the upper boundary of the region is given by f and the lower bound is given by g. Therefore the area of the region is the value of the integral

$$\begin{aligned}\int_0^{4\pi} \big(f(x) - g(x)\big)\,dx &= \int_0^{4\pi} \Big(\sin x + 2 - \big(\frac{1}{2}\cos(2x) - 1\big)\Big)\,dx \\ &= -\cos x - \frac{1}{4}\sin(2x) + 3x\Big|_0^{4\pi} \\ &= 12\pi \approx 37.7 \text{ units}^2.\end{aligned}$$

Example 7.1.2 Finding total area enclosed by curves

Find the total area of the region enclosed by the functions $f(x) = -2x + 5$ and $g(x) = x^3 - 7x^2 + 12x - 3$ as shown in Figure 7.1.3.

Solution A quick calculation shows that $f = g$ at $x = 1, 2$ and 4. One can proceed thoughtlessly by computing $\int_1^4 \big(f(x) - g(x)\big)\,dx$, but this ignores the fact that on $[1, 2]$, $g(x) > f(x)$. (In fact, the thoughtless integration returns $-9/4$, hardly the expected value of an *area*.) Thus we compute the total area by breaking the interval $[1, 4]$ into two subintervals, $[1, 2]$ and $[2, 4]$ and using the proper integrand in each.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \big(g(x) - f(x)\big)\,dx + \int_2^4 \big(f(x) - g(x)\big)\,dx \\ &= \int_1^2 \big(x^3 - 7x^2 + 14x - 8\big)\,dx + \int_2^4 \big(-x^3 + 7x^2 - 14x + 8\big)\,dx \\ &= 5/12 + 8/3 \\ &= 37/12 = 3.083 \text{ units}^2.\end{aligned}$$

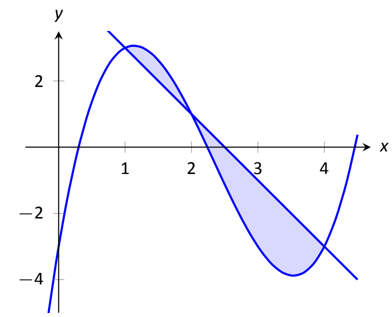


Figure 7.1.3: Graphing a region enclosed by two functions in Example 7.1.2.

The previous example makes note that we are expecting area to be *positive*. When first learning about the definite integral, we interpreted it as "signed area under the curve," allowing for "negative area." That doesn't apply here; area is to be positive.

The previous example also demonstrates that we often have to break a given region into subregions before applying Theorem 7.1.1. The following example shows another situation where this is applicable, along with an alternate view of applying the Theorem.

Example 7.1.3 Finding area: integrating with respect to *y*

Find the area of the region enclosed by the functions $y = \sqrt{x} + 2$, $y = -(x-1)^2 + 3$ and $y = 2$, as shown in Figure 7.1.4.

Solution We give two approaches to this problem. In the first approach, we notice that the region's "top" is defined by two different curves. On $[0, 1]$, the top function is $y = \sqrt{x} + 2$; on $[1, 2]$, the top function is $y = -(x-1)^2 + 3$. Thus we compute the area as the sum of two integrals:

$$\begin{aligned}\text{Total Area} &= \int_0^1 \Big(\big(\sqrt{x} + 2\big) - 2\Big)\,dx + \int_1^2 \Big(\big(-(x-1)^2 + 3\big) - 2\Big)\,dx \\ &= 2/3 + 2/3 \\ &= 4/3.\end{aligned}$$

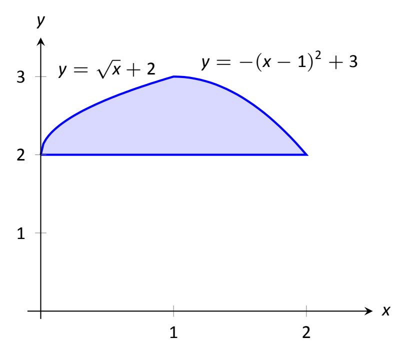


Figure 7.1.4: Graphing a region for Example 7.1.3.

The second approach is clever and very useful in certain situations. We are used to viewing curves as functions of x; we input an x-value and a y-value is returned. Some curves can also be described as functions of y: input a y-value and an x-value is returned. We can rewrite the equations describing the boundary by solving for x:

$$\begin{aligned} y &= \sqrt{x} + 2 &\Rightarrow\quad x &= (y-2)^2 \\ y &= -(x-1)^2 + 3 &\Rightarrow\quad x &= \sqrt{3-y} + 1.\end{aligned}$$

Figure 7.1.5 shows the region with the boundaries relabelled. A differential element, a horizontal rectangle, is also pictured. The width of the rectangle is

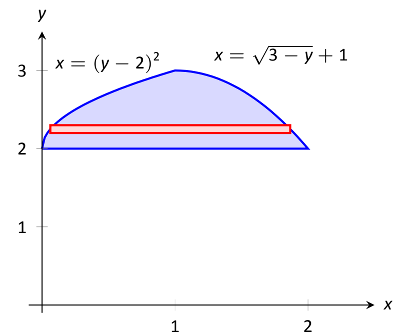


Figure 7.1.5: The region used in Example 7.1.3 with boundaries relabelled as functions of y.

a small change in y: Δy. The height of the rectangle is a difference in x-values. The "top" x-value is the largest value, i.e., the rightmost. The "bottom" x-value is the smaller, i.e., the leftmost. Therefore the height of the rectangle is

$$\big(\sqrt{3-y}+1\big)-(y-2)^2.$$

The area is found by integrating the above function with respect to y with the appropriate bounds. We determine these by considering the y-values the region occupies. It is bounded below by $y=2$, and bounded above by $y=3$. That is, both the "top" and "bottom" functions exist on the y interval $[2,3]$. Thus

$$\begin{aligned}\text{Total Area} &= \int_2^3 \big(\sqrt{3-y}+1-(y-2)^2\big)\,dy\\ &= \Big(-\frac{2}{3}(3-y)^{3/2}+y-\frac{1}{3}(y-2)^3\Big)\Big|_2^3\\ &= 4/3.\end{aligned}$$

This calculus–based technique of finding area can be useful even with shapes that we normally think of as "easy." Example 7.1.4 computes the area of a triangle. While the formula "$\frac{1}{2}\times$ base $\times$ height" is well known, in arbitrary triangles it can be nontrivial to compute the height. Calculus makes the problem simple.

Example 7.1.4 Finding the area of a triangle

Compute the area of the regions bounded by the lines $y=x+1$, $y=-2x+7$ and $y=-\frac{1}{2}x+\frac{5}{2}$, as shown in Figure 7.1.6.

SOLUTION Recognize that there are two "top" functions to this region, causing us to use two definite integrals.

$$\begin{aligned}\text{Total Area} &= \int_1^2 \Big((x+1)-(-\frac{1}{2}x+\frac{5}{2})\Big)\,dx+\int_2^3 \Big((-2x+7)-(-\frac{1}{2}x+\frac{5}{2})\Big)\,dx\\ &= 3/4+3/4\\ &= 3/2.\end{aligned}$$

We can also approach this by converting each function into a function of y. This also requires 2 integrals, so there isn't really any advantage to doing so. We do it here for demonstration purposes.

The "top" function is always $x=\frac{7-y}{2}$ while there are two "bottom" functions. Being mindful of the proper integration bounds, we have

$$\begin{aligned}\text{Total Area} &= \int_1^2 \Big(\frac{7-y}{2}-(5-2y)\Big)\,dy+\int_2^3 \Big(\frac{7-y}{2}-(y-1)\Big)\,dy\\ &= 3/4+3/4\\ &= 3/2.\end{aligned}$$

Of course, the final answer is the same. (It is interesting to note that the area of all 4 subregions used is 3/4. This is coincidental.)

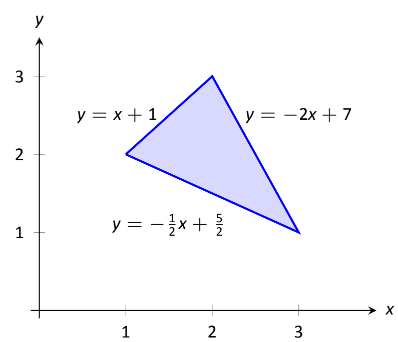


Figure 7.1.6: Graphing a triangular region in Example 7.1.4.

In the next section we apply our applications–of–integration techniques to finding the volumes of certain solids.

Exercises 7.1

Terms and Concepts

1. T/F: The area between curves is always positive.

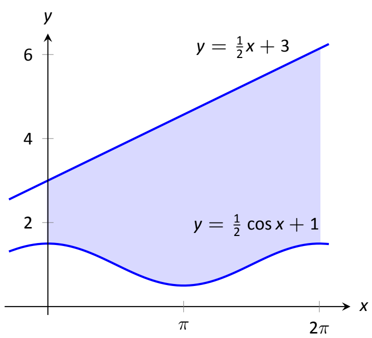
2. T/F: Calculus can be used to find the area of basic geometric shapes.

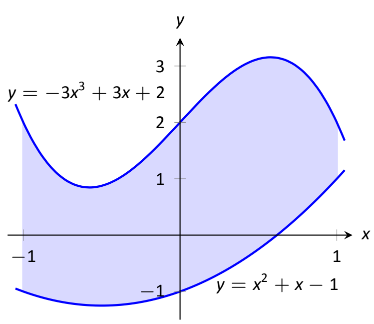
3. In your own words, describe how to find the total area enclosed by $y = f(x)$ and $y = g(x)$.

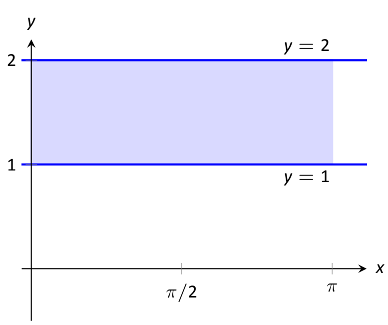
4. Describe a situation where it is advantageous to find an area enclosed by curves through integration with respect to y instead of x.

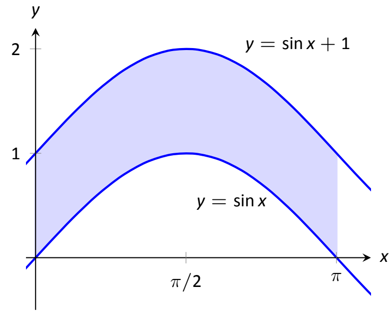
Problems

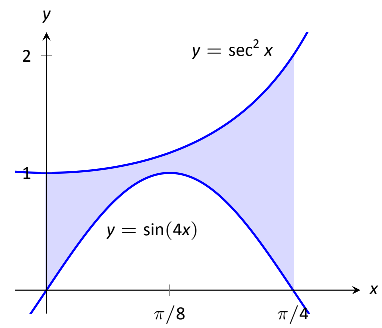
In Exercises 5 – 12, find the area of the shaded region in the given graph.

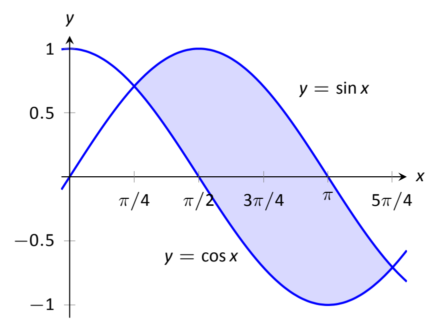
5. 

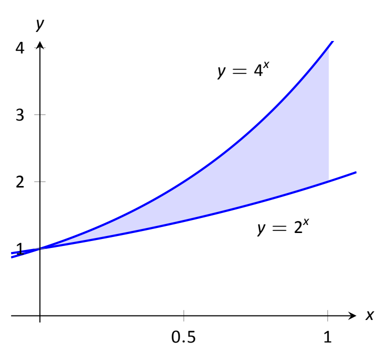
6. 

7. 

8. 

9. 

10. 

11. 

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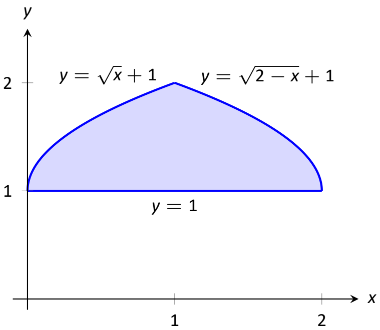


In Exercises 13 – 20, find the total area enclosed by the functions f and g.

13. $f(x)=2x^2+5x-3$, $g(x)=x^2+4x-1$

14. $f(x)=x^2-3x+2$, $g(x)=-3x+3$

15. $f(x)=\sin x$, $g(x)=2x/\pi$

16. $f(x)=x^3-4x^2+x-1$, $g(x)=-x^2+2x-4$

17. $f(x)=x$, $g(x)=\sqrt{x}$

18. $f(x)=-x^3+5x^2+2x+1$, $g(x)=3x^2+x+3$

19. The functions $f(x)=\cos(x)$ and $g(x)=\sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

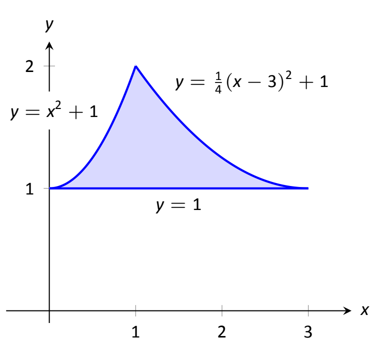
20. The functions $f(x)=\cos(2x)$ and $g(x)=\sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

21. The functions $f(x)=\cos(2x)$ and $g(x)=\sin x$ intersect infinitely many times, forming an infinite number of repeated, enclosed regions. Find the areas of these regions.

In Exercises 22 – 27, find the area of the enclosed region in two ways:

1. **by treating the boundaries as functions of x, and**
2. **by treating the boundaries as functions of y.**

22.



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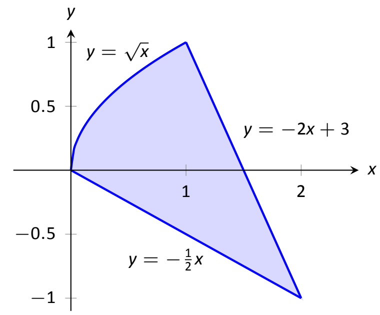

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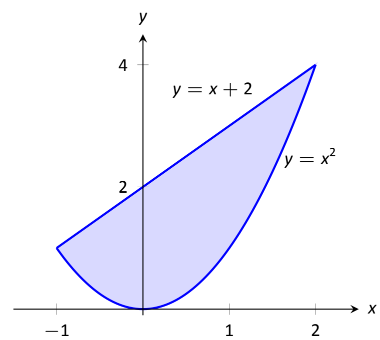

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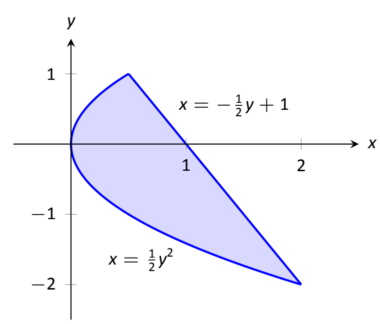

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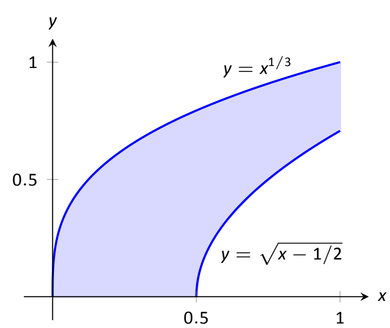

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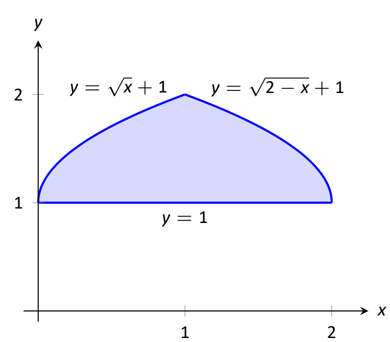


In Exercises 28 – 31, find the area triangle formed by the given three points.

28. $(1, 1)$, $(2, 3)$, and $(3, 3)$

29. $(-1, 1)$, $(1, 3)$, and $(2, -1)$

30. $(1, 1)$, $(3, 3)$, and $(3, 3)$

31. $(0, 0)$, $(2, 5)$, and $(5, 2)$

7.2 Volume by Cross-Sectional Area; Disk and Washer Methods

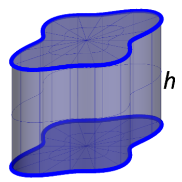


Figure 7.2.1: The volume of a general right cylinder

The volume of a general right cylinder, as shown in Figure 7.2.1, is

Area of the base × height.

We can use this fact as the building block in finding volumes of a variety of shapes.

Given an arbitrary solid, we can *approximate* its volume by cutting it into n thin slices. When the slices are thin, each slice can be approximated well by a general right cylinder. Thus the volume of each slice is approximately its cross-sectional area × thickness. (These slices are the differential elements.)

By orienting a solid along the x-axis, we can let $A(x_i)$ represent the cross-sectional area of the i^{th} slice, and let Δx_i represent the thickness of this slice (the thickness is a small change in x). The total volume of the solid is approximately:

$$\begin{aligned}\text{Volume} &\approx \sum_{i=1}^{n} \Big[\text{Area} \times \text{thickness}\Big]\\ &= \sum_{i=1}^{n} A(x_i)\Delta x_i.\end{aligned}$$

Recognize that this is a Riemann Sum. By taking a limit (as the thickness of the slices goes to 0) we can find the volume exactly.

Theorem 7.2.1 Volume By Cross-Sectional Area

The volume V of a solid, oriented along the x-axis with cross-sectional area $A(x)$ from $x = a$ to $x = b$, is

$$V = \int_a^b A(x)\,dx.$$

Example 7.2.1 Finding the volume of a solid

Find the volume of a pyramid with a square base of side length 10 in and a height of 5 in.

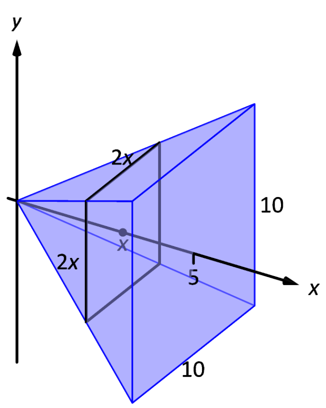


Figure 7.2.2: Orienting a pyramid along the x-axis in Example 7.2.1.

Solution There are many ways to "orient" the pyramid along the x-axis; Figure 7.2.2 gives one such way, with the pointed top of the pyramid at the origin and the x-axis going through the center of the base.

Each cross section of the pyramid is a square; this is a sample differential element. To determine its area $A(x)$, we need to determine the side lengths of the square.

When $x = 5$, the square has side length 10; when $x = 0$, the square has side length 0. Since the edges of the pyramid are lines, it is easy to figure that each cross-sectional square has side length $2x$, giving $A(x) = (2x)^2 = 4x^2$.

If one were to cut a slice out of the pyramid at $x = 3$, as shown in Figure 7.2.3, one would have a shape with square bottom and top with sloped sides. If the slice were thin, both the bottom and top squares would have sides lengths of about 6, and thus the cross–sectional area of the bottom and top would be about 36in^2. Letting Δx_i represent the thickness of the slice, the volume of this slice would then be about $36\Delta x_i \text{in}^3$.

Cutting the pyramid into n slices divides the total volume into n equally–spaced smaller pieces, each with volume $(2x_i)^2\Delta x$, where x_i is the approximate location of the slice along the x-axis and Δx represents the thickness of each slice. One can approximate total volume of the pyramid by summing up the volumes of these slices:

$$\text{Approximate volume } = \sum_{i=1}^{n} (2x_i)^2 \Delta x.$$

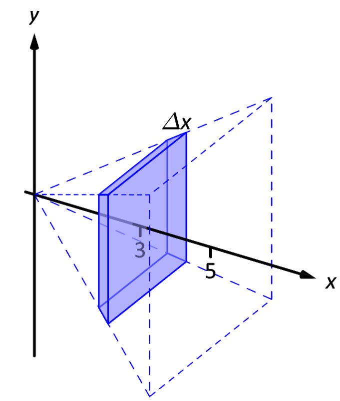


Figure 7.2.3: Cutting a slice in they pyramid in Example 7.2.1 at $x = 3$.

Taking the limit as $n \to \infty$ gives the actual volume of the pyramid; recoginizing this sum as a Riemann Sum allows us to find the exact answer using a definite integral, matching the definite integral given by Theorem 7.2.1.

We have

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} (2x_i)^2 \Delta x \\ &= \int_0^5 4x^2 \, dx \\ &= \frac{4}{3}x^3 \Big|_0^5 \\ &= \frac{500}{3} \text{ in}^3 \approx 166.67 \text{ in}^3. \end{aligned}$$

We can check our work by consulting the general equation for the volume of a pyramid (see the back cover under "Volume of A General Cone"):

$$\tfrac{1}{3} \times \text{area of base} \times \text{height}.$$

Certainly, using this formula from geometry is faster than our new method, but the calculus–based method can be applied to much more than just cones.

An important special case of Theorem 7.2.1 is when the solid is a **solid of revolution**, that is, when the solid is formed by rotating a shape around an axis.

Start with a function $y = f(x)$ from $x = a$ to $x = b$. Revolving this curve about a horizontal axis creates a three-dimensional solid whose cross sections are disks (thin circles). Let $R(x)$ represent the radius of the cross-sectional disk at x; the area of this disk is $\pi R(x)^2$. Applying Theorem 7.2.1 gives the Disk Method.

Key Idea 7.2.1 The Disk Method

Let a solid be formed by revolving the curve $y = f(x)$ from $x = a$ to $x = b$ around a horizontal axis, and let $R(x)$ be the radius of the cross-sectional disk at x. The volume of the solid is

$$V = \pi \int_a^b R(x)^2 \, dx.$$

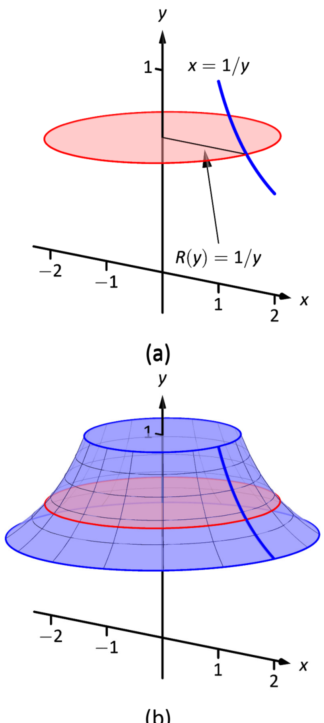


Figure 7.2.4: Sketching a solid in Example 7.2.2.

Figure 7.2.5: Sketching a solid in Example 7.2.3.

Example 7.2.2 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, around the x-axis.

Solution A sketch can help us understand this problem. In Figure 7.2.4(a) the curve $y = 1/x$ is sketched along with the differential element – a disk – at x with radius $R(x) = 1/x$. In Figure 7.2.4 (b) the whole solid is pictured, along with the differential element.

The volume of the differential element shown in part (a) of the figure is approximately $\pi R(x_i)^2\Delta x$, where $R(x_i)$ is the radius of the disk shown and Δx is the thickness of that slice. The radius $R(x_i)$ is the distance from the x-axis to the curve, hence $R(x_i) = 1/x_i$.

Slicing the solid into n equally–spaced slices, we can approximate the total volume by adding up the approximate volume of each slice:

$$\text{Approximate volume } = \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x.$$

Taking the limit of the above sum as $n \to \infty$ gives the actual volume; recognizing this sum as a Riemann sum allows us to evaluate the limit with a definite integral, which matches the formula given in Key Idea 7.2.1:

$$\begin{aligned} V &= \lim_{n\to\infty} \sum_{i=1}^{n} \pi \left(\frac{1}{x_i}\right)^2 \Delta x \\ &= \pi \int_1^2 \left(\frac{1}{x}\right)^2 dx \\ &= \pi \int_1^2 \frac{1}{x^2}\, dx \\ &= \pi \left[-\frac{1}{x}\right]\Big|_1^2 \\ &= \pi\left[-\frac{1}{2} - (-1)\right] \\ &= \frac{\pi}{2} \text{ units}^3. \end{aligned}$$

While Key Idea 7.2.1 is given in terms of functions of x, the principle involved can be applied to functions of y when the axis of rotation is vertical, not horizontal. We demonstrate this in the next example.

Example 7.2.3 Finding volume using the Disk Method

Find the volume of the solid formed by revolving the curve $y = 1/x$, from $x = 1$ to $x = 2$, about the y-axis.

Solution Since the axis of rotation is vertical, we need to convert the function into a function of y and convert the x-bounds to y-bounds. Since $y = 1/x$ defines the curve, we rewrite it as $x = 1/y$. The bound $x = 1$ corresponds to the y-bound $y = 1$, and the bound $x = 2$ corresponds to the y-bound $y = 1/2$.

Thus we are rotating the curve $x = 1/y$, from $y = 1/2$ to $y = 1$ about the y-axis to form a solid. The curve and sample differential element are sketched in Figure 7.2.5 (a), with a full sketch of the solid in Figure 7.2.5 (b). We integrate to find the volume:

$$\begin{aligned} V &= \pi \int_{1/2}^{1} \frac{1}{y^2}\, dy \\ &= -\frac{\pi}{y}\Big|_{1/2}^{1} \\ &= \pi \text{ units}^3. \end{aligned}$$

We can also compute the volume of solids of revolution that have a hole in the center. The general principle is simple: compute the volume of the solid irrespective of the hole, then subtract the volume of the hole. If the outside radius of the solid is $R(x)$ and the inside radius (defining the hole) is $r(x)$, then the volume is

$$V = \pi \int_a^b R(x)^2\, dx - \pi \int_a^b r(x)^2\, dx = \pi \int_a^b \big(R(x)^2 - r(x)^2\big)\, dx.$$

One can generate a solid of revolution with a hole in the middle by revolving a region about an axis. Consider Figure 7.2.6(a), where a region is sketched along with a dashed, horizontal axis of rotation. By rotating the region about the axis, a solid is formed as sketched in Figure 7.2.6(b). The outside of the solid has radius $R(x)$, whereas the inside has radius $r(x)$. Each cross section of this solid will be a washer (a disk with a hole in the center) as sketched in Figure 7.2.7. This leads us to the Washer Method.

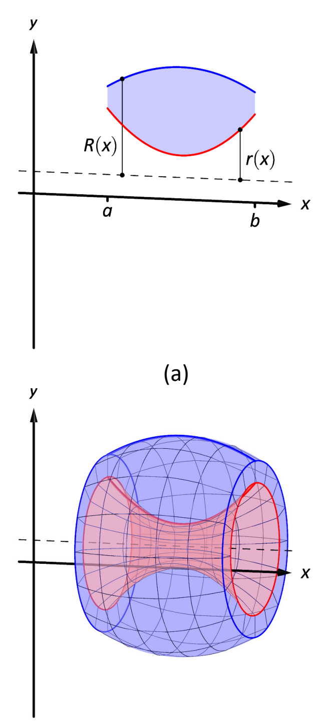


Figure 7.2.6: Establishing the Washer Method; see also Figure 7.2.7.

Key Idea 7.2.2 The Washer Method

Let a region bounded by $y = f(x)$, $y = g(x)$, $x = a$ and $x = b$ be rotated about a horizontal axis that does not intersect the region, forming a solid. Each cross section at x will be a washer with outside radius $R(x)$ and inside radius $r(x)$. The volume of the solid is

$$V = \pi \int_a^b \Big(R(x)^2 - r(x)^2\Big)\, dx.$$

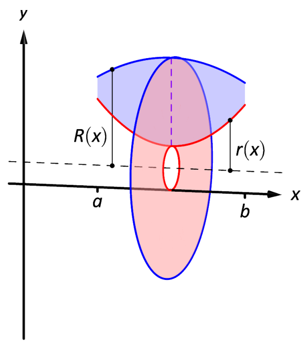


Figure 7.2.7: Establishing the Washer Method; see also Figure 7.2.6.

Even though we introduced it first, the Disk Method is just a special case of the Washer Method with an inside radius of $r(x) = 0$.

Example 7.2.4 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$ about the x-axis.

SOLUTION A sketch of the region will help, as given in Figure 7.2.8(a). Rotating about the x-axis will produce cross sections in the shape of washers, as shown in Figure 7.2.8(b); the complete solid is shown in part (c). The outside radius of this washer is $R(x) = 2x + 1$; the inside radius is $r(x) = x^2 - 2x + 2$. As the region is bounded from $x = 1$ to $x = 3$, we integrate as follows to compute the volume.

$$\begin{aligned} V &= \pi \int_1^3 \Big((2x-1)^2 - (x^2-2x+2)^2\Big)\,dx \\ &= \pi \int_1^3 \Big(-x^4 + 4x^3 - 4x^2 + 4x - 3\Big)\,dx \\ &= \pi \Big[-\frac{1}{5}x^5 + x^4 - \frac{4}{3}x^3 + 2x^2 - 3x\Big]\Big|_1^3 \\ &= \frac{104}{15}\pi \approx 21.78 \text{ units}^3. \end{aligned}$$

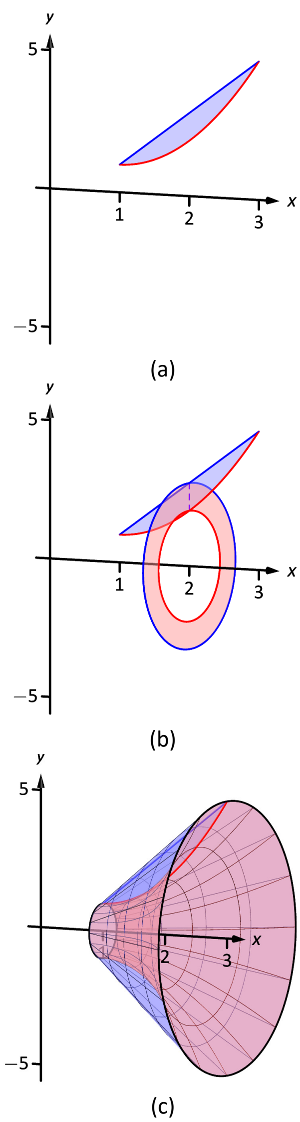


Figure 7.2.8: Sketching the differential element and solid in Example 7.2.4.

When rotating about a vertical axis, the outside and inside radius functions must be functions of y.

Example 7.2.5 Finding volume with the Washer Method

Find the volume of the solid formed by rotating the triangular region with vertices at $(1, 1)$, $(2, 1)$ and $(2, 3)$ about the y-axis.

SOLUTION The triangular region is sketched in Figure 7.2.9(a); the differential element is sketched in (b) and the full solid is drawn in (c). They help us establish the outside and inside radii. Since the axis of rotation is vertical, each radius is a function of y.

The outside radius $R(y)$ is formed by the line connecting $(2, 1)$ and $(2, 3)$; it is a constant function, as regardless of the y-value the distance from the line to the axis of rotation is 2. Thus $R(y) = 2$.

The inside radius is formed by the line connecting $(1, 1)$ and $(2, 3)$. The equation of this line is $y = 2x - 1$, but we need to refer to it as a function of y. Solving for x gives $r(y) = \frac{1}{2}(y + 1)$.

We integrate over the y-bounds of $y = 1$ to $y = 3$. Thus the volume is

$$\begin{aligned} V &= \pi \int_1^3 \Big(2^2 - \big(\tfrac{1}{2}(y+1)\big)^2\Big)\,dy \\ &= \pi \int_1^3 \Big(-\frac{1}{4}y^2 - \frac{1}{2}y + \frac{15}{4}\Big)\,dy \\ &= \pi \Big[-\frac{1}{12}y^3 - \frac{1}{4}y^2 + \frac{15}{4}y\Big]\Big|_1^3 \\ &= \frac{10}{3}\pi \approx 10.47 \text{ units}^3. \end{aligned}$$

This section introduced a new application of the definite integral. Our default view of the definite integral is that it gives "the area under the curve." However, we can establish definite integrals that represent other quantities; in this section, we computed volume.

The ultimate goal of this section is not to compute volumes of solids. That can be useful, but what is more useful is the understanding of this basic principle of integral calculus, outlined in Key Idea 7.0.1: to find the exact value of some quantity,

- we start with an approximation (in this section, slice the solid and approximate the volume of each slice),
- then make the approximation better by refining our original approximation (i.e., use more slices),
- then use limits to establish a definite integral which gives the exact value.

We practice this principle in the next section where we find volumes by slicing solids in a different way.

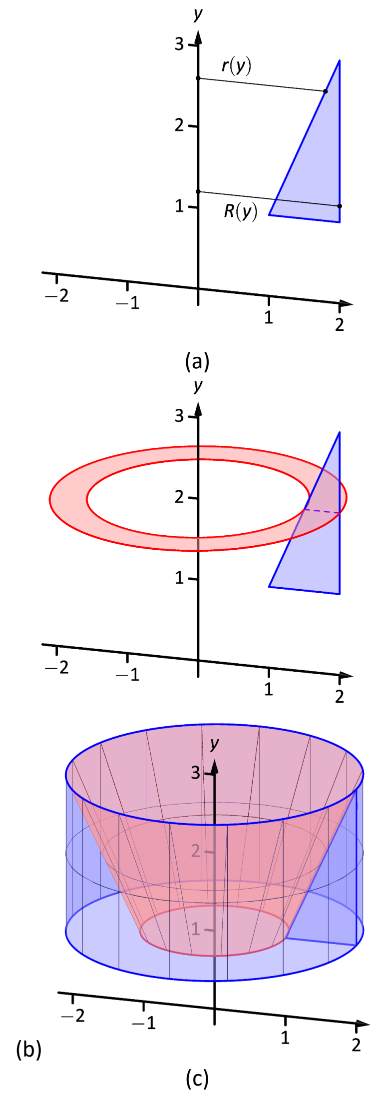


Figure 7.2.9: Sketching the solid in Example 7.2.5.

Exercises 7.2

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

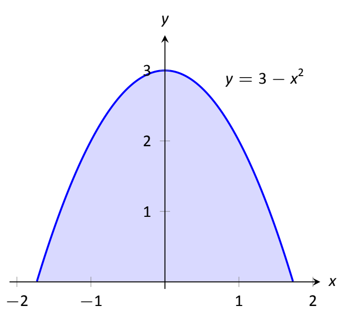
2. In your own words, explain how the Disk and Washer Methods are related.

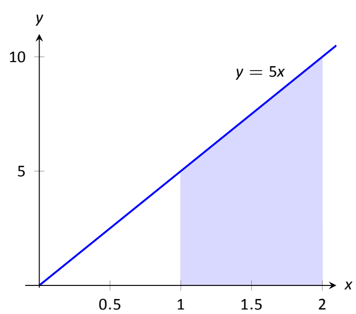
3. Explain the how the units of volume are found in the integral of Theorem 7.2.1: if $A(x)$ has units of in^2, how does $\int A(x)\,dx$ have units of in^3?

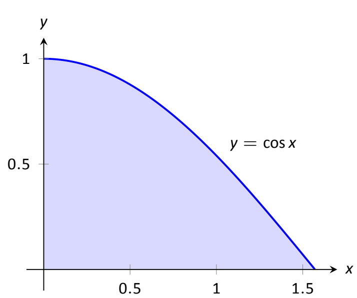
4. A fundamental principle of this section is "_________ can be found by integrating an area function."

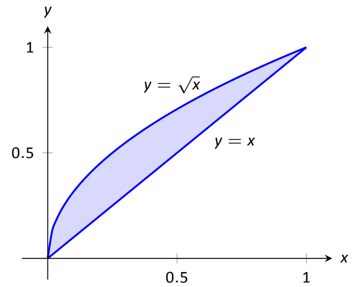
Problems

In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the *x*-axis.

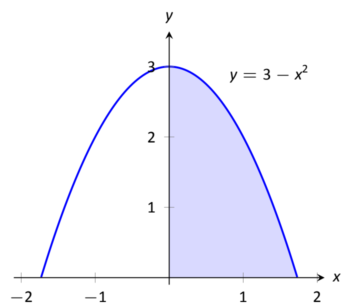
5. 

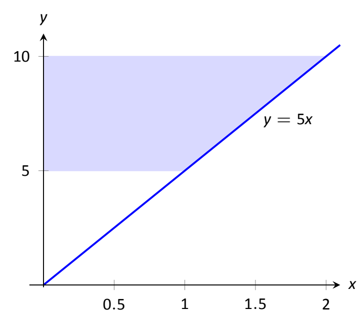
6. 

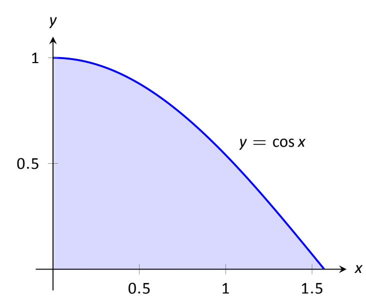
7. 

8. 

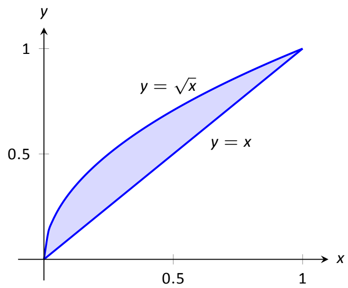
In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Disk/Washer Method to find the volume of the solid of revolution formed by revolving the region about the *y*-axis.

9. 

10. 

11. 

(Hint: Integration By Parts will be necessary, twice. First let $u = \arccos^2 x$, then let $u = \arccos x$.)

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Disk/Washer Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:

(a) the x-axis
(b) $y = 1$
(c) the y-axis
(d) $x = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:

(a) the x-axis
(b) $y = 4$
(c) $y = -1$
(d) $x = 2$

15. The triangle with vertices $(1, 1)$, $(1, 2)$ and $(2, 1)$.
Rotate about:

(a) the x-axis
(b) $y = 2$
(c) the y-axis
(d) $x = 1$

16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:

(a) the x-axis
(b) $y = 1$
(c) $y = 5$

17. Region bounded by $y = 1/\sqrt{x^2 + 1}$, $x = -1$, $x = 1$ and the x-axis.
Rotate about:

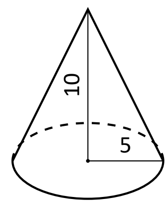
(a) the x-axis
(b) $y = 1$
(c) $y = -1$

18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:

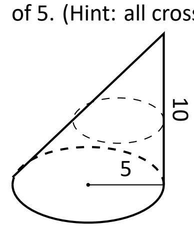
(a) the x-axis
(b) $y = 4$
(c) the y-axis
(d) $x = 2$

In Exercises 19 – 22, a solid is described. Orient the solid along the x-axis such that a cross-sectional area function $A(x)$ can be obtained, then apply Theorem 7.2.1 to find the volume of the solid.

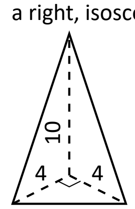
19. A right circular cone with height of 10 and base radius of 5.



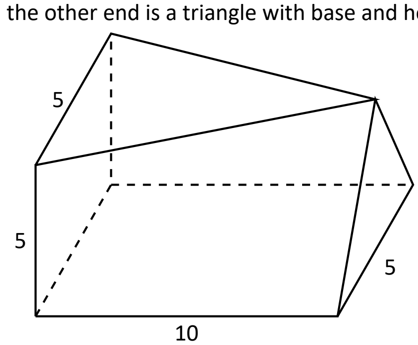
20. A skew right circular cone with height of 10 and base radius of 5. (Hint: all cross-sections are circles.)



21. A right triangular cone with height of 10 and whose base is a right, isosceles triangle with side length 4.



22. A solid with length 10 with a rectangular base and triangular top, wherein one end is a square with side length 5 and the other end is a triangle with base and height of 5.



7.3 The Shell Method

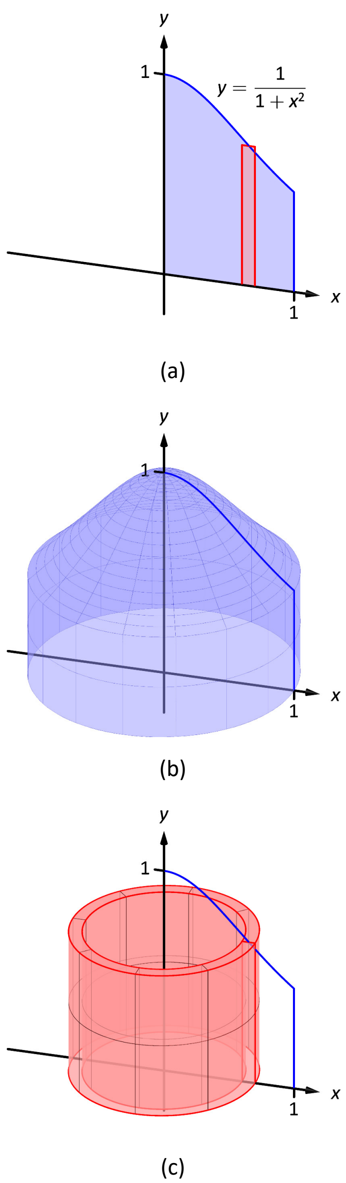


Figure 7.3.1: Introducing the Shell Method.

Often a given problem can be solved in more than one way. A particular method may be chosen out of convenience, personal preference, or perhaps necessity. Ultimately, it is good to have options.

The previous section introduced the Disk and Washer Methods, which computed the volume of solids of revolution by integrating the cross–sectional area of the solid. This section develops another method of computing volume, the **Shell Method.** Instead of slicing the solid perpendicular to the axis of rotation creating cross-sections, we now slice it parallel to the axis of rotation, creating "shells."

Consider Figure 7.3.1, where the region shown in (a) is rotated around the y-axis forming the solid shown in (b). A small slice of the region is drawn in (a), parallel to the axis of rotation. When the region is rotated, this thin slice forms a **cylindrical shell**, as pictured in part (c) of the figure. The previous section approximated a solid with lots of thin disks (or washers); we now approximate a solid with many thin cylindrical shells.

To compute the volume of one shell, first consider the paper label on a soup can with radius r and height h. What is the area of this label? A simple way of determining this is to cut the label and lay it out flat, forming a rectangle with height h and length $2\pi r$. Thus the area is $A = 2\pi rh$; see Figure 7.3.2(a).

Do a similar process with a cylindrical shell, with height h, thickness Δx, and approximate radius r. Cutting the shell and laying it flat forms a rectangular solid with length $2\pi r$, height h and depth Δx. Thus the volume is $V \approx 2\pi rh\Delta x$; see Figure 7.3.2(b). (We say "approximately" since our radius was an approximation.)

By breaking the solid into n cylindrical shells, we can approximate the volume of the solid as

$$V = \sum_{i=1}^{n} 2\pi r_i h_i \Delta x_i,$$

where r_i, h_i and Δx_i are the radius, height and thickness of the i^{th} shell, respectively.

This is a Riemann Sum. Taking a limit as the thickness of the shells approaches 0 leads to a definite integral.

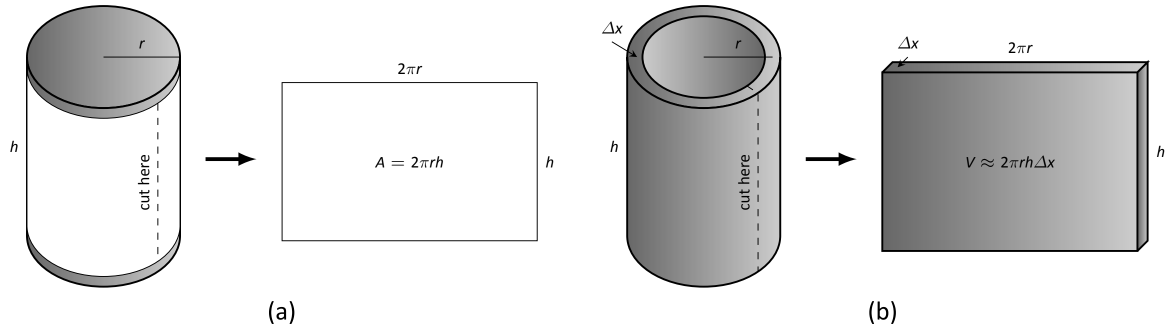


Figure 7.3.2: Determining the volume of a thin cylindrical shell.

> **Key Idea 7.3.1 The Shell Method**
>
> Let a solid be formed by revolving a region R, bounded by $x = a$ and $x = b$, around a vertical axis. Let $r(x)$ represent the distance from the axis of rotation to x (i.e., the radius of a sample shell) and let $h(x)$ represent the height of the solid at x (i.e., the height of the shell). The volume of the solid is
>
> $$V = 2\pi \int_a^b r(x)h(x)\,dx.$$

Special Cases:

1. When the region R is bounded above by $y = f(x)$ and below by $y = g(x)$, then $h(x) = f(x) - g(x)$.

2. When the axis of rotation is the y-axis (i.e., $x = 0$) then $r(x) = x$.

Let's practice using the Shell Method.

Example 7.3.1 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region bounded by $y = 0$, $y = 1/(1 + x^2)$, $x = 0$ and $x = 1$ about the y-axis.

Solution This is the region used to introduce the Shell Method in Figure 7.3.1, but is sketched again in Figure 7.3.3 for closer reference. A line is drawn in the region parallel to the axis of rotation representing a shell that will be carved out as the region is rotated about the y-axis. (This is the differential element.)

The distance this line is from the axis of rotation determines $r(x)$; as the distance from x to the y-axis is x, we have $r(x) = x$. The height of this line determines $h(x)$; the top of the line is at $y = 1/(1 + x^2)$, whereas the bottom of the line is at $y = 0$. Thus $h(x) = 1/(1 + x^2) - 0 = 1/(1 + x^2)$. The region is bounded from $x = 0$ to $x = 1$, so the volume is

$$V = 2\pi \int_0^1 \frac{x}{1 + x^2}\,dx.$$

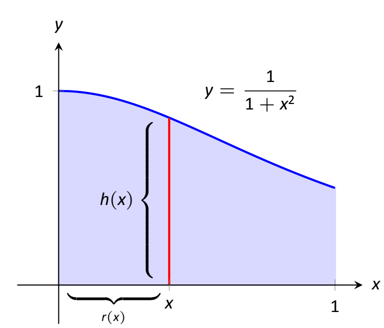


Figure 7.3.3: Graphing a region in Example 7.3.1.

This requires substitution. Let $u = 1 + x^2$, so $du = 2x\,dx$. We also change the bounds: $u(0) = 1$ and $u(1) = 2$. Thus we have:

$$= \pi \int_1^2 \frac{1}{u}\,du$$
$$= \pi \ln u \Big|_1^2$$
$$= \pi \ln 2 \approx 2.178 \text{ units}^3.$$

Note: in order to find this volume using the Disk Method, two integrals would be needed to account for the regions above and below $y = 1/2$.

With the Shell Method, nothing special needs to be accounted for to compute the volume of a solid that has a hole in the middle, as demonstrated next.

Example 7.3.2 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the triangular region determined by the points $(0,1)$, $(1,1)$ and $(1,3)$ about the line $x = 3$.

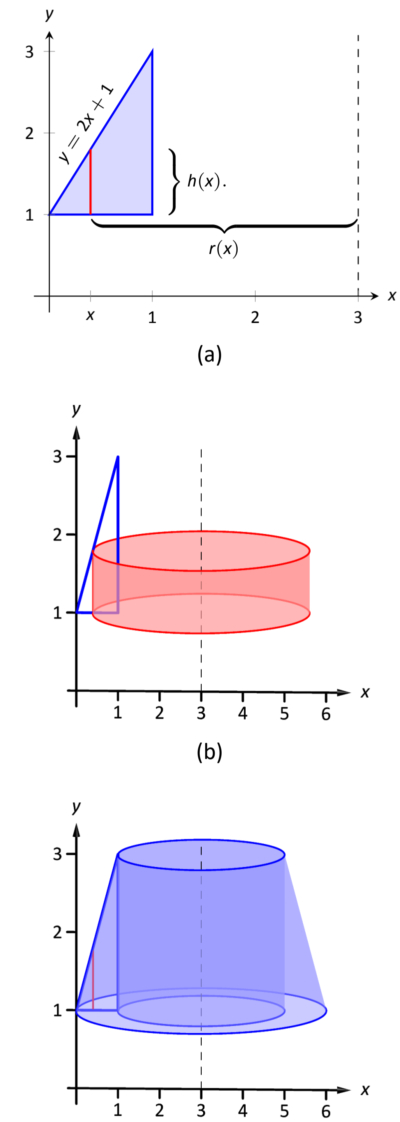


Figure 7.3.4: Graphing a region in Example 7.3.2.

Solution The region is sketched in Figure 7.3.4(a) along with the differential element, a line within the region parallel to the axis of rotation. In part (b) of the figure, we see the shell traced out by the differential element, and in part (c) the whole solid is shown.

The height of the differential element is the distance from $y = 1$ to $y = 2x + 1$, the line that connects the points $(0,1)$ and $(1,3)$. Thus $h(x) = 2x + 1 - 1 = 2x$. The radius of the shell formed by the differential element is the distance from x to $x = 3$; that is, it is $r(x) = 3 - x$. The x-bounds of the region are $x = 0$ to $x = 1$, giving

$$V = 2\pi \int_0^1 (3 - x)(2x)\,dx$$
$$= 2\pi \int_0^1 \left(6x - 2x^2\right) dx$$
$$= 2\pi \left(3x^2 - \frac{2}{3}x^3\right)\Big|_0^1$$
$$= \frac{14}{3}\pi \approx 14.66 \text{ units}^3.$$

When revolving a region around a horizontal axis, we must consider the radius and height functions in terms of y, not x.

Example 7.3.3 Finding volume using the Shell Method

Find the volume of the solid formed by rotating the region given in Example 7.3.2 about the x-axis.

Solution The region is sketched in Figure 7.3.5(a) with a sample differential element. In part (b) of the figure the shell formed by the differential element is drawn, and the solid is sketched in (c). (Note that the triangular region looks "short and wide" here, whereas in the previous example the same region looked "tall and narrow." This is because the bounds on the graphs are different.)

The height of the differential element is an x-distance, between $x = \frac{1}{2}y - \frac{1}{2}$ and $x = 1$. Thus $h(y) = 1 - (\frac{1}{2}y - \frac{1}{2}) = -\frac{1}{2}y + \frac{3}{2}$. The radius is the distance from

y to the x-axis, so $r(y) = y$. The y bounds of the region are $y = 1$ and $y = 3$, leading to the integral

$$\begin{aligned} V &= 2\pi \int_1^3 \left[y \left(-\frac{1}{2}y + \frac{3}{2} \right) \right] dy \\ &= 2\pi \int_1^3 \left[-\frac{1}{2}y^2 + \frac{3}{2}y \right] dy \\ &= 2\pi \left[-\frac{1}{6}y^3 + \frac{3}{4}y^2 \right] \Bigg|_1^3 \\ &= 2\pi \left[\frac{9}{4} - \frac{7}{12} \right] \\ &= \frac{10}{3}\pi \approx 10.472 \text{ units}^3. \end{aligned}$$

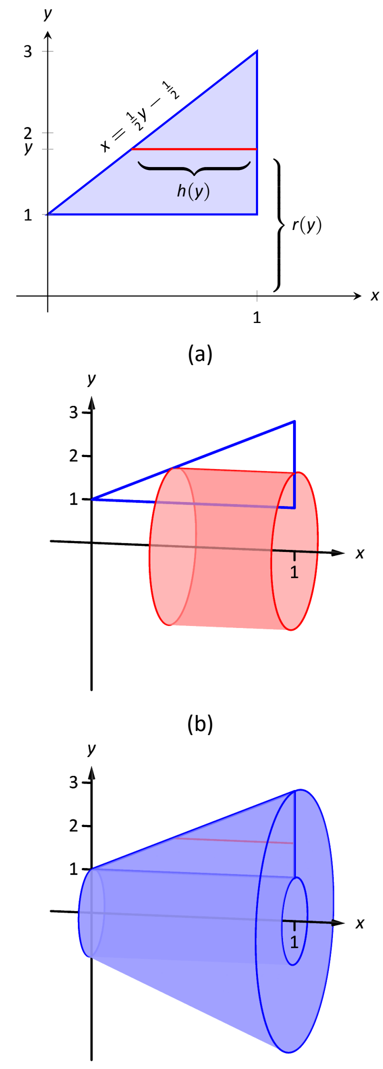


Figure 7.3.5: Graphing a region in Example 7.3.3.

At the beginning of this section it was stated that "it is good to have options." The next example finds the volume of a solid rather easily with the Shell Method, but using the Washer Method would be quite a chore.

Example 7.3.4 Finding volume using the Shell Method

Find the volume of the solid formed by revolving the region bounded by $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ about the y-axis.

Solution The region and a differential element, the shell formed by this differential element, and the resulting solid are given in Figure 7.3.6.

The radius of a sample shell is $r(x) = x$; the height of a sample shell is $h(x) = \sin x$, each from $x = 0$ to $x = \pi$. Thus the volume of the solid is

$$V = 2\pi \int_0^\pi x \sin x \, dx.$$

This requires Integration By Parts. Set $u = x$ and $dv = \sin x \, dx$; we leave it to the reader to fill in the rest. We have:

$$\begin{aligned} &= 2\pi \left[-x\cos x \Big|_0^\pi + \int_0^\pi \cos x \, dx \right] \\ &= 2\pi \left[\pi + \sin x \Big|_0^\pi \right] \\ &= 2\pi \left[\pi + 0 \right] \\ &= 2\pi^2 \approx 19.74 \text{ units}^3. \end{aligned}$$

Note that in order to use the Washer Method, we would need to solve $y = \sin x$ for x, requiring the use of the arcsine function. We leave it to the reader to verify that the outside radius function is $R(y) = \pi - \arcsin y$ and the inside radius function is $r(y) = \arcsin y$. Thus the volume can be computed as

$$\pi \int_0^1 \left[(\pi - \arcsin y)^2 - (\arcsin y)^2 \right] dy.$$

This integral isn't terrible given that the $\arcsin^2 y$ terms cancel, but it is more onerous than the integral created by the Shell Method.

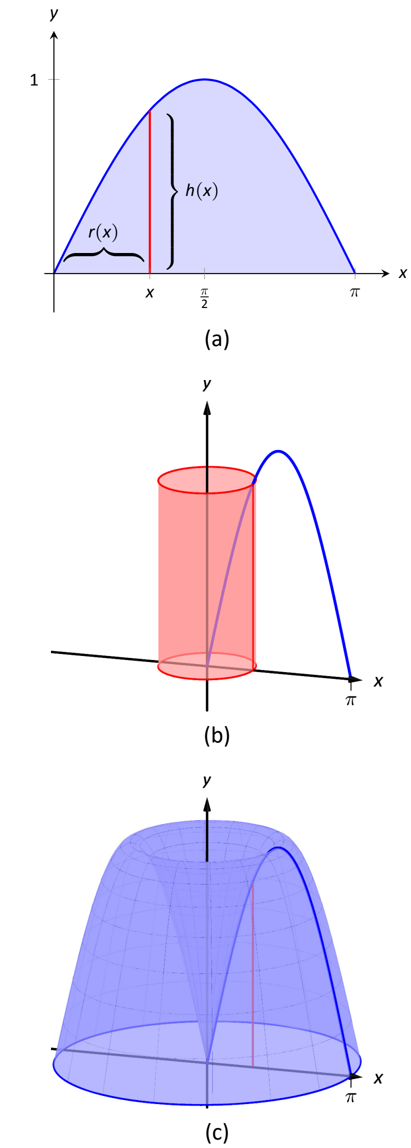


Figure 7.3.6: Graphing a region in Example 7.3.4.

We end this section with a table summarizing the usage of the Washer and Shell Methods.

Key Idea 7.3.2 Summary of the Washer and Shell Methods

Let a region R be given with x-bounds $x = a$ and $x = b$ and y-bounds $y = c$ and $y = d$.

	Washer Method	Shell Method
Horizontal Axis	$\pi \int_a^b \left(R(x)^2 - r(x)^2\right)\,dx$	$2\pi \int_c^d r(y)h(y)\,dy$
Vertical Axis	$\pi \int_c^d \left(R(y)^2 - r(y)^2\right)\,dy$	$2\pi \int_a^b r(x)h(x)\,dx$

As in the previous section, the real goal of this section is not to be able to compute volumes of certain solids. Rather, it is to be able to solve a problem by first approximating, then using limits to refine the approximation to give the exact value. In this section, we approximate the volume of a solid by cutting it into thin cylindrical shells. By summing up the volumes of each shell, we get an approximation of the volume. By taking a limit as the number of equally spaced shells goes to infinity, our summation can be evaluated as a definite integral, giving the exact value. We use this same principle again in the next section, where we find the length of curves in the plane.

Exercises 7.3

Terms and Concepts

1. T/F: A solid of revolution is formed by revolving a shape around an axis.

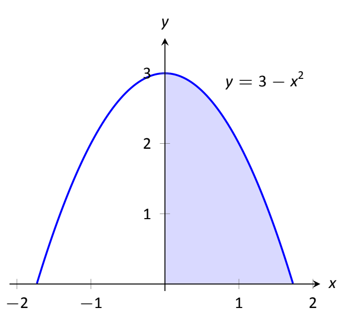
2. T/F: The Shell Method can only be used when the Washer Method fails.

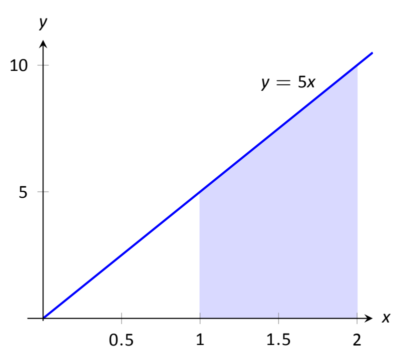
3. T/F: The Shell Method works by integrating cross–sectional areas of a solid.

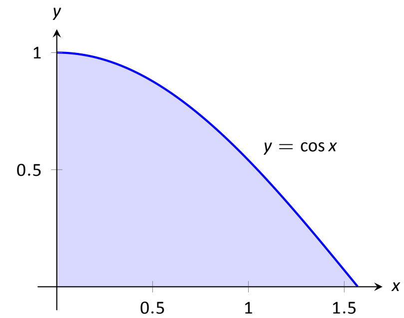
4. T/F: When finding the volume of a solid of revolution that was revolved around a vertical axis, the Shell Method integrates with respect to x.

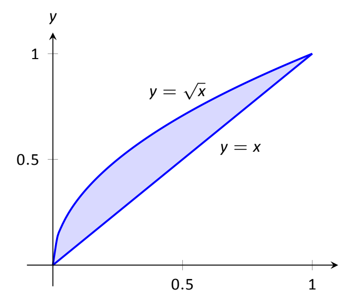
Problems

In Exercises 5 – 8, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the y-axis.

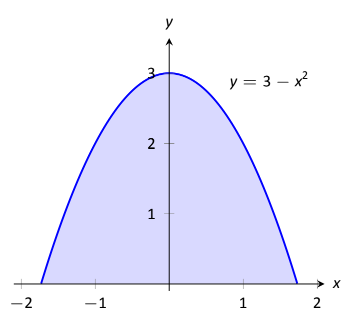
5. 

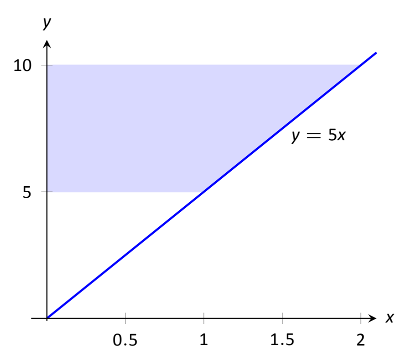
6. 

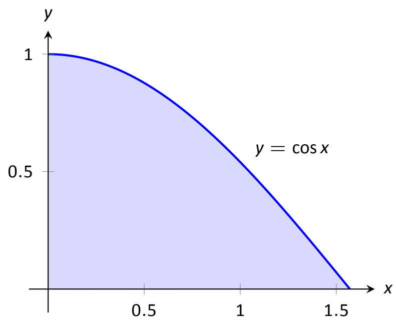
7. 

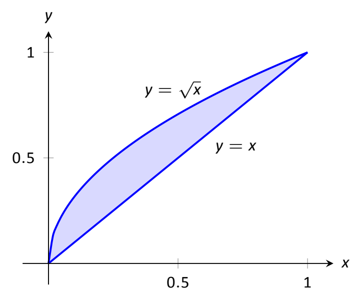
8. 

In Exercises 9 – 12, a region of the Cartesian plane is shaded. Use the Shell Method to find the volume of the solid of revolution formed by revolving the region about the x-axis.

9. 

10. 

11. 

12. 

In Exercises 13 – 18, a region of the Cartesian plane is described. Use the Shell Method to find the volume of the solid of revolution formed by rotating the region about each of the given axes.

13. Region bounded by: $y = \sqrt{x}$, $y = 0$ and $x = 1$.
Rotate about:
(a) the y-axis
(b) $x = 1$
(c) the x-axis
(d) $y = 1$

14. Region bounded by: $y = 4 - x^2$ and $y = 0$.
Rotate about:
(a) $x = 2$
(b) $x = -2$
(c) the x-axis
(d) $y = 4$

15. The triangle with vertices $(1,1)$, $(1,2)$ and $(2,1)$.
Rotate about:
(a) the y-axis
(b) $x = 1$
(c) the x-axis
(d) $y = 2$

16. Region bounded by $y = x^2 - 2x + 2$ and $y = 2x - 1$.
Rotate about:
(a) the y-axis
(b) $x = 1$
(c) $x = -1$

17. Region bounded by $y = 1/\sqrt{x^2+1}$, $x = 1$ and the x and y-axes.
Rotate about:
(a) the y-axis
(b) $x = 1$

18. Region bounded by $y = 2x$, $y = x$ and $x = 2$.
Rotate about:
(a) the y-axis
(b) $x = 2$
(c) the x-axis
(d) $y = 4$

7.4 Arc Length and Surface Area

In previous sections we have used integration to answer the following questions:

1. Given a region, what is its area?
2. Given a solid, what is its volume?

In this section, we address a related question: Given a curve, what is its length? This is often referred to as **arc length**.

Consider the graph of $y = \sin x$ on $[0, \pi]$ given in Figure 7.4.1(a). How long is this curve? That is, if we were to use a piece of string to exactly match the shape of this curve, how long would the string be?

As we have done in the past, we start by approximating; later, we will refine our answer using limits to get an exact solution.

The length of straight–line segments is easy to compute using the Distance Formula. We can approximate the length of the given curve by approximating the curve with straight lines and measuring their lengths.

In Figure 7.4.1(b), the curve $y = \sin x$ has been approximated with 4 line segments (the interval $[0, \pi]$ has been divided into 4 equally–lengthed subintervals). It is clear that these four line segments approximate $y = \sin x$ very well on the first and last subinterval, though not so well in the middle. Regardless, the sum of the lengths of the line segments is 3.79, so we approximate the arc length of $y = \sin x$ on $[0, \pi]$ to be 3.79.

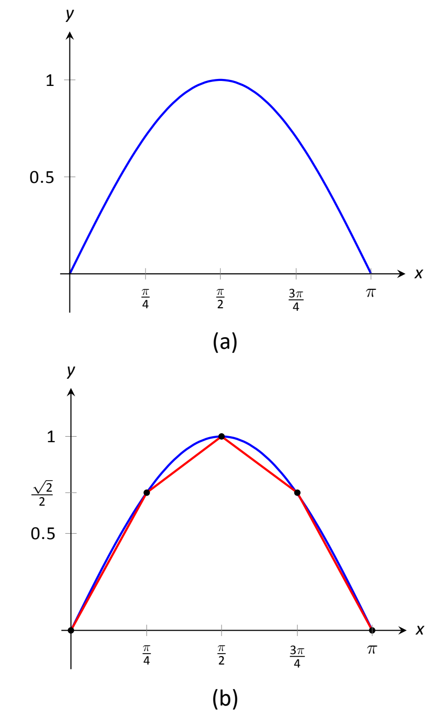


Figure 7.4.1: Graphing $y = \sin x$ on $[0, \pi]$ and approximating the curve with line segments.

In general, we can approximate the arc length of $y = f(x)$ on $[a, b]$ in the following manner. Let $a = x_1 < x_2 < \ldots < x_n < x_{n+1} = b$ be a partition of $[a, b]$ into n subintervals. Let Δx_i represent the length of the i^{th} subinterval $[x_i, x_{i+1}]$.

Figure 7.4.2 zooms in on the i^{th} subinterval where $y = f(x)$ is approximated by a straight line segment. The dashed lines show that we can view this line segment as the hypotenuse of a right triangle whose sides have length Δx_i and Δy_i. Using the Pythagorean Theorem, the length of this line segment is $\sqrt{\Delta x_i^2 + \Delta y_i^2}$. Summing over all subintervals gives an arc length approximation

$$L \approx \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

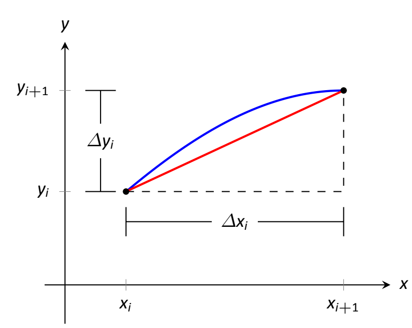


Figure 7.4.2: Zooming in on the i^{th} subinterval $[x_i, x_{i+1}]$ of a partition of $[a, b]$.

As shown here, this is *not* a Riemann Sum. While we could conclude that taking a limit as the subinterval length goes to zero gives the exact arc length, we would not be able to compute the answer with a definite integral. We need first to do a little algebra.

In the above expression factor out a Δx_i^2 term:

$$\sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 \left(1 + \frac{\Delta y_i^2}{\Delta x_i^2}\right)}.$$

Now pull the Δx_i^2 term out of the square root:

$$= \sum_{i=1}^{n} \sqrt{1 + \frac{\Delta y_i^2}{\Delta x_i^2}}\, \Delta x_i.$$

This is nearly a Riemann Sum. Consider the $\Delta y_i^2/\Delta x_i^2$ term. The expression $\Delta y_i/\Delta x_i$ measures the "change in y/change in x," that is, the "rise over run" of f on the i^{th} subinterval. The Mean Value Theorem of Differentiation (Theorem 3.2.1) states that there is a c_i in the i^{th} subinterval where $f'(c_i) = \Delta y_i/\Delta x_i$. Thus we can rewrite our above expression as:

$$= \sum_{i=1}^{n} \sqrt{1+f'(c_i)^2}\,\Delta x_i.$$

This *is* a Riemann Sum. As long as f' is continuous, we can invoke Theorem 5.3.2 and conclude

$$= \int_a^b \sqrt{1+f'(x)^2}\,dx.$$

Theorem 7.4.1 Arc Length

Let f be differentiable on $[a,b]$, where f' is also continuous on $[a,b]$. Then the arc length of f from $x=a$ to $x=b$ is

$$L = \int_a^b \sqrt{1+f'(x)^2}\,dx.$$

Note: This is our first use of differentiability on a closed interval since Section 2.1.

The theorem also requires that f' be continuous on $[a,b]$; while examples are arcane, it is possible for f to be differentiable yet f' is not continuous.

As the integrand contains a square root, it is often difficult to use the formula in Theorem 7.4.1 to find the length exactly. When exact answers are difficult to come by, we resort to using numerical methods of approximating definite integrals. The following examples will demonstrate this.

Example 7.4.1 Finding arc length

Find the arc length of $f(x) = x^{3/2}$ from $x=0$ to $x=4$.

Solution We find $f'(x) = \frac{3}{2}x^{1/2}$; note that on $[0,4]$, f is differentiable and f' is also continuous. Using the formula, we find the arc length L as

$$\begin{aligned} L &= \int_0^4 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx \\ &= \int_0^4 \sqrt{1+\frac{9}{4}x}\,dx \\ &= \int_0^4 \left(1+\frac{9}{4}x\right)^{1/2}\,dx \\ &= \frac{2}{3}\cdot\frac{4}{9}\cdot\left(1+\frac{9}{4}x\right)^{3/2}\Bigg|_0^4 \\ &= \frac{8}{27}\left(10^{3/2}-1\right) \approx 9.07\text{units}. \end{aligned}$$

A graph of f is given in Figure 7.4.3.

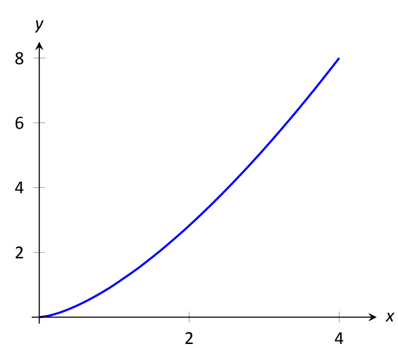


Figure 7.4.3: A graph of $f(x) = x^{3/2}$ from Example 7.4.1.

Example 7.4.2 Finding arc length

Find the arc length of $f(x) = \frac{1}{8}x^2 - \ln x$ from $x=1$ to $x=2$.

Solution This function was chosen specifically because the resulting integral can be evaluated exactly. We begin by finding $f'(x) = x/4 - 1/x$. The arc length is

$$\begin{aligned} L &= \int_1^2 \sqrt{1+\left(\frac{x}{4}-\frac{1}{x}\right)^2}\,dx \\ &= \int_1^2 \sqrt{1+\frac{x^2}{16}-\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\frac{x^2}{16}+\frac{1}{2}+\frac{1}{x^2}}\,dx \\ &= \int_1^2 \sqrt{\left(\frac{x}{4}+\frac{1}{x}\right)^2}\,dx \\ &= \int_1^2 \left(\frac{x}{4}+\frac{1}{x}\right)\,dx \\ &= \left(\frac{x^2}{8}+\ln x\right)\Bigg|_1^2 \\ &= \frac{3}{8}+\ln 2 \approx 1.07 \text{ units.} \end{aligned}$$

A graph of f is given in Figure 7.4.4; the portion of the curve measured in this problem is in bold.

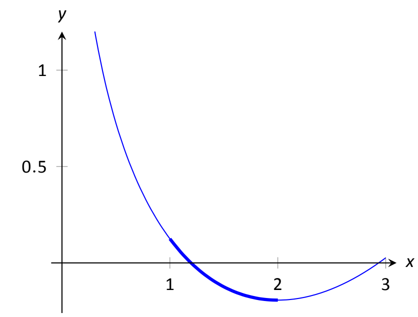


Figure 7.4.4: A graph of $f(x) = \frac{1}{8}x^2 - \ln x$ from Example 7.4.2.

The previous examples found the arc length exactly through careful choice of the functions. In general, exact answers are much more difficult to come by and numerical approximations are necessary.

Example 7.4.3 Approximating arc length numerically

Find the length of the sine curve from $x = 0$ to $x = \pi$.

Solution This is somewhat of a mathematical curiosity; in Example 5.4.3 we found the area under one "hump" of the sine curve is 2 square units; now we are measuring its arc length.

The setup is straightforward: $f(x) = \sin x$ and $f'(x) = \cos x$. Thus

$$L = \int_0^\pi \sqrt{1+\cos^2 x}\,dx.$$

This integral *cannot* be evaluated in terms of elementary functions so we will approximate it with Simpson's Method with $n = 4$. Figure 7.4.5 gives $\sqrt{1+\cos^2 x}$ evaluated at 5 evenly spaced points in $[0, \pi]$. Simpson's Rule then states that

$$\begin{aligned} \int_0^\pi \sqrt{1+\cos^2 x}\,dx &\approx \frac{\pi-0}{4\cdot 3}\left(\sqrt{2}+4\sqrt{3/2}+2(1)+4\sqrt{3/2}+\sqrt{2}\right) \\ &= 3.82918. \end{aligned}$$

Using a computer with $n = 100$ the approximation is $L \approx 3.8202$; our approximation with $n = 4$ is quite good.

x	$\sqrt{1+\cos^2 x}$
0	$\sqrt{2}$
$\pi/4$	$\sqrt{3/2}$
$\pi/2$	1
$3\pi/4$	$\sqrt{3/2}$
π	$\sqrt{2}$

Figure 7.4.5: A table of values of $y = \sqrt{1+\cos^2 x}$ to evaluate a definite integral in Example 7.4.3.

Surface Area of Solids of Revolution

We have already seen how a curve $y = f(x)$ on $[a, b]$ can be revolved around an axis to form a solid. Instead of computing its volume, we now consider its surface area.

We begin as we have in the previous sections: we partition the interval $[a, b]$ with n subintervals, where the i^{th} subinterval is $[x_i, x_{i+1}]$. On each subinterval, we can approximate the curve $y = f(x)$ with a straight line that connects $f(x_i)$ and $f(x_{i+1})$ as shown in Figure 7.4.6(a). Revolving this line segment about the x-axis creates part of a cone (called a *frustum* of a cone) as shown in Figure 7.4.6(b). The surface area of a frustum of a cone is

$$2\pi \cdot \text{ length } \cdot \text{average of the two radii } R \text{ and } r.$$

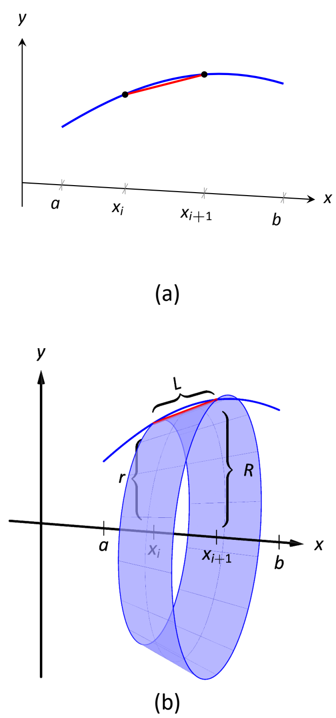


Figure 7.4.6: Establishing the formula for surface area.

The length is given by L; we use the material just covered by arc length to state that

$$L \approx \sqrt{1 + f'(c_i)}\Delta x_i$$

for some c_i in the i^{th} subinterval. The radii are just the function evaluated at the endpoints of the interval. That is,

$$R = f(x_{i+1}) \quad \text{and} \quad r = f(x_i).$$

Thus the surface area of this sample frustum of the cone is approximately

$$2\pi\frac{f(x_i) + f(x_{i+1})}{2}\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

Since f is a continuous function, the Intermediate Value Theorem states there is some d_i in $[x_i, x_{i+1}]$ such that $f(d_i) = \dfrac{f(x_i) + f(x_{i+1})}{2}$; we can use this to rewrite the above equation as

$$2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i.$$

Summing over all the subintervals we get the total surface area to be approximately

$$\text{Surface Area} \approx \sum_{i=1}^{n} 2\pi f(d_i)\sqrt{1 + f'(c_i)^2}\Delta x_i,$$

which is a Riemann Sum. Taking the limit as the subinterval lengths go to zero gives us the exact surface area, given in the following Key Idea.

Theorem 7.4.2 Surface Area of a Solid of Revolution

Let f be differentiable on $[a, b]$, where f' is also continuous on $[a, b]$.

1. The surface area of the solid formed by revolving the graph of $y = f(x)$, where $f(x) \geq 0$, about the x-axis is

$$\text{Surface Area} = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\, dx.$$

2. The surface area of the solid formed by revolving the graph of $y = f(x)$ about the y-axis, where $a, b \geq 0$, is

$$\text{Surface Area} = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\, dx.$$

(When revolving $y = f(x)$ about the y-axis, the radii of the resulting frustum are x_i and x_{i+1}; their average value is simply the midpoint of the interval. In the limit, this midpoint is just x. This gives the second part of Theorem 7.4.2.)

Example 7.4.4 Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving $y = \sin x$ on $[0, \pi]$ around the x-axis, as shown in Figure 7.4.7.

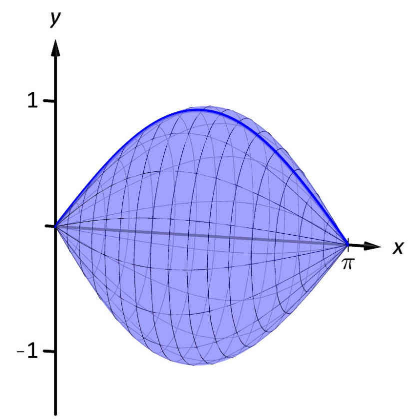


Figure 7.4.7: Revolving $y = \sin x$ on $[0, \pi]$ about the x-axis.

Solution The setup is relatively straightforward. Using Theorem 7.4.2, we have the surface area SA is:

$$\begin{aligned} SA &= 2\pi \int_0^{\pi} \sin x \sqrt{1+\cos^2 x}\, dx \\ &= -2\pi \frac{1}{2}\left(\sinh^{-1}(\cos x) + \cos x\sqrt{1+\cos^2 x}\right)\Big|_0^{\pi} \\ &= 2\pi\left(\sqrt{2} + \sinh^{-1} 1\right) \approx 14.42 \text{ units}^2. \end{aligned}$$

The integration step above is nontrivial, utilizing an integration method called Trigonometric Substitution.

It is interesting to see that the surface area of a solid, whose shape is defined by a trigonometric function, involves both a square root and an inverse hyperbolic trigonometric function.

Example 7.4.5 Finding surface area of a solid of revolution

Find the surface area of the solid formed by revolving the curve $y = x^2$ on $[0, 1]$ about the x-axis and the y-axis.

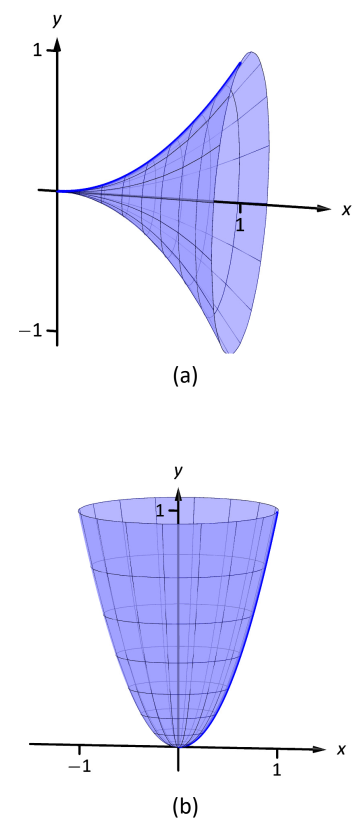


Figure 7.4.8: The solids used in Example 7.4.5.

Solution About the x-axis: the integral is straightforward to setup:

$$SA = 2\pi \int_0^1 x^2\sqrt{1+(2x)^2}\, dx.$$

Like the integral in Example 7.4.4, this requires Trigonometric Substitution.

$$\begin{aligned} &= \frac{\pi}{32}\left(2(8x^3+x)\sqrt{1+4x^2} - \sinh^{-1}(2x)\right)\Big|_0^1 \\ &= \frac{\pi}{32}\left(18\sqrt{5} - \sinh^{-1} 2\right) \\ &\approx 3.81 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ around the x-axis is graphed in Figure 7.4.8 (a).

About the y-axis: since we are revolving around the y-axis, the "radius" of the solid is not $f(x)$ but rather x. Thus the integral to compute the surface area is:

$$SA = 2\pi \int_0^1 x\sqrt{1+(2x)^2}\, dx.$$

This integral can be solved using substitution. Set $u = 1 + 4x^2$; the new bounds are $u = 1$ to $u = 5$. We then have

$$\begin{aligned} &= \frac{\pi}{4}\int_1^5 \sqrt{u}\,du \\ &= \frac{\pi}{4}\frac{2}{3}u^{3/2}\Big|_1^5 \\ &= \frac{\pi}{6}\left(5\sqrt{5}-1\right) \\ &\approx 5.33 \text{ units}^2. \end{aligned}$$

The solid formed by revolving $y = x^2$ about the y-axis is graphed in Figure 7.4.8 (b).

Our final example is a famous mathematical "paradox."

Example 7.4.6 The surface area and volume of Gabriel's Horn

Consider the solid formed by revolving $y = 1/x$ about the x-axis on $[1, \infty)$. Find the volume and surface area of this solid. (This shape, as graphed in Figure 7.4.9, is known as "Gabriel's Horn" since it looks like a very long horn that only a supernatural person, such as an angel, could play.)

Solution To compute the volume it is natural to use the Disk Method. We have:

$$\begin{aligned} V &= \pi\int_1^\infty \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi\int_1^b \frac{1}{x^2}\,dx \\ &= \lim_{b\to\infty} \pi\left(\frac{-1}{x}\right)\Big|_1^b \\ &= \lim_{b\to\infty} \pi\left(1-\frac{1}{b}\right) \\ &= \pi \text{ units}^3. \end{aligned}$$

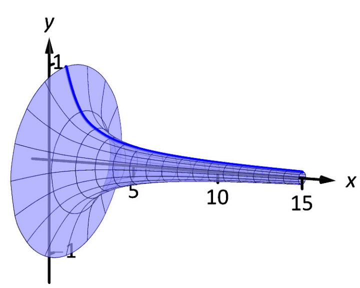


Figure 7.4.9: A graph of Gabriel's Horn.

Gabriel's Horn has a finite volume of π cubic units. Since we have already seen that regions with infinite length can have a finite area, this is not too difficult to accept.

We now consider its surface area. The integral is straightforward to setup:

$$SA = 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

Integrating this expression is not trivial. We can, however, compare it to other improper integrals. Since $1 < \sqrt{1+1/x^4}$ on $[1, \infty)$, we can state that

$$2\pi\int_1^\infty \frac{1}{x}\,dx < 2\pi\int_1^\infty \frac{1}{x}\sqrt{1+1/x^4}\,dx.$$

By Key Idea 6.8.1, the improper integral on the left diverges. Since the integral on the right is larger, we conclude it also diverges, meaning Gabriel's Horn has infinite surface area.

Hence the "paradox": we can fill Gabriel's Horn with a finite amount of paint, but since it has infinite surface area, we can never paint it.

Somehow this paradox is striking when we think about it in terms of volume and area. However, we have seen a similar paradox before, as referenced above. We know that the area under the curve $y = 1/x^2$ on $[1, \infty)$ is finite, yet the shape has an infinite perimeter. Strange things can occur when we deal with the infinite.

A standard equation from physics is "Work = force $\times$ distance", when the force applied is constant. In the next section we learn how to compute work when the force applied is variable.

Exercises 7.4

Terms and Concepts

1. T/F: The integral formula for computing Arc Length was found by first approximating arc length with straight line segments.

2. T/F: The integral formula for computing Arc Length includes a square–root, meaning the integration is probably easy.

Problems

In Exercises 3 – 12, find the arc length of the function on the given interval.

3. $f(x) = x$ on $[0, 1]$.

4. $f(x) = \sqrt{8}x$ on $[-1, 1]$.

5. $f(x) = \dfrac{1}{3}x^{3/2} - x^{1/2}$ on $[0, 1]$.

6. $f(x) = \dfrac{1}{12}x^3 + \dfrac{1}{x}$ on $[1, 4]$.

7. $f(x) = 2x^{3/2} - \dfrac{1}{6}\sqrt{x}$ on $[0, 9]$.

8. $f(x) = \cosh x$ on $[-\ln 2, \ln 2]$.

9. $f(x) = \dfrac{1}{2}\left(e^x + e^{-x}\right)$ on $[0, \ln 5]$.

10. $f(x) = \dfrac{1}{12}x^5 + \dfrac{1}{5x^3}$ on $[.1, 1]$.

11. $f(x) = \ln\left(\sin x\right)$ on $[\pi/6, \pi/2]$.

12. $f(x) = \ln\left(\cos x\right)$ on $[0, \pi/4]$.

In Exercises 13 – 20, set up the integral to compute the arc length of the function on the given interval. Do not evaluate the integral.

13. $f(x) = x^2$ on $[0, 1]$.

14. $f(x) = x^{10}$ on $[0, 1]$.

15. $f(x) = \sqrt{x}$ on $[0, 1]$.

16. $f(x) = \ln x$ on $[1, e]$.

17. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: this describes the top half of a circle with radius 1.)

18. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: this describes the top half of an ellipse with a major axis of length 6 and a minor axis of length 2.)

19. $f(x) = \dfrac{1}{x}$ on $[1, 2]$.

20. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 21 – 28, use Simpson's Rule, with $n = 4$, to approximate the arc length of the function on the given interval. Note: these are the same problems as in Exercises 13–20.

21. $f(x) = x^2$ on $[0, 1]$.

22. $f(x) = x^{10}$ on $[0, 1]$.

23. $f(x) = \sqrt{x}$ on $[0, 1]$. (Note: $f'(x)$ is not defined at $x = 0$.)

24. $f(x) = \ln x$ on $[1, e]$.

25. $f(x) = \sqrt{1 - x^2}$ on $[-1, 1]$. (Note: $f'(x)$ is not defined at the endpoints.)

26. $f(x) = \sqrt{1 - x^2/9}$ on $[-3, 3]$. (Note: $f'(x)$ is not defined at the endpoints.)

27. $f(x) = \dfrac{1}{x}$ on $[1, 2]$.

28. $f(x) = \sec x$ on $[-\pi/4, \pi/4]$.

In Exercises 29 – 33, find the surface area of the described solid of revolution.

29. The solid formed by revolving $y = 2x$ on $[0, 1]$ about the x-axis.

30. The solid formed by revolving $y = x^2$ on $[0, 1]$ about the y-axis.

31. The solid formed by revolving $y = x^3$ on $[0, 1]$ about the x-axis.

32. The solid formed by revolving $y = \sqrt{x}$ on $[0, 1]$ about the x-axis.

33. The sphere formed by revolving $y = \sqrt{1 - x^2}$ on $[-1, 1]$ about the x-axis.

7.5 Work

Work is the scientific term used to describe the action of a force which moves an object. When a constant force F is applied to move an object a distance d, the amount of work performed is $W = F \cdot d$.

The SI unit of force is the Newton, (kg·m/s^2), and the SI unit of distance is a metre (m). The fundamental unit of work is one Newton–metre, or a joule (J). That is, applying a force of one Newton for one metre performs one joule of work. In Imperial units (as used in the United States), force is measured in pounds (lb) and distance is measured in feet (ft), hence work is measured in ft–lb.

When force is constant, the measurement of work is straightforward. For instance, lifting a 200 lb object 5 ft performs $200 \cdot 5 = 1000$ ft–lb of work.

What if the force applied is variable? For instance, imagine a climber pulling a 200 ft rope up a vertical face. The rope becomes lighter as more is pulled in, requiring less force and hence the climber performs less work.

In general, let $F(x)$ be a force function on an interval $[a, b]$. We want to measure the amount of work done applying the force F from $x = a$ to $x = b$. We can approximate the amount of work being done by partitioning $[a, b]$ into subintervals $a = x_1 < x_2 < \cdots < x_{n+1} = b$ and assuming that F is constant on each subinterval. Let c_i be a value in the i^{th} subinterval $[x_i, x_{i+1}]$. Then the work done on this interval is approximately $W_i \approx F(c_i) \cdot (x_{i+1} - x_i) = F(c_i)\Delta x_i$, a constant force $\times$ the distance over which it is applied. The total work is

$$W = \sum_{i=1}^{n} W_i \approx \sum_{i=1}^{n} F(c_i)\Delta x_i.$$

This, of course, is a Riemann sum. Taking a limit as the subinterval lengths go to zero give an exact value of work which can be evaluated through a definite integral.

Key Idea 7.5.1 Work

Let $F(x)$ be a continuous function on $[a, b]$ describing the amount of force being applied to an object in the direction of travel from distance $x = a$ to distance $x = b$. The total work W done on $[a, b]$ is

$$W = \int_a^b F(x)\, dx.$$

Note: *Mass* and *weight* are closely related, yet different, concepts. The mass m of an object is a quantitative measure of that object's resistance to acceleration. The weight w of an object is a measurement of the force applied to the object by the acceleration of gravity g.

Since the two measurements are proportional, $w = m \cdot g$, they are often used interchangeably in everyday conversation. When computing work, one must be careful to note which is being referred to. When mass is given, it must be multiplied by the acceleration of gravity to reference the related force.

Example 7.5.1 Computing work performed: applying variable force

A 60m climbing rope is hanging over the side of a tall cliff. How much work is performed in pulling the rope up to the top, where the rope has a mass of 66g/m?

Solution We need to create a force function $F(x)$ on the interval $[0, 60]$. To do so, we must first decide what x is measuring: it is the length of the rope still hanging or is it the amount of rope pulled in? As long as we are consistent, either approach is fine. We adopt for this example the convention that x is the amount of rope pulled in. This seems to match intuition better; pulling up the first 10 meters of rope involves $x = 0$ to $x = 10$ instead of $x = 60$ to $x = 50$.

As x is the amount of rope pulled in, the amount of rope still hanging is $60-x$. This length of rope has a mass of 66 g/m, or 0.066 kg/m. The mass of the rope still hanging is $0.066(60 - x)$ kg; multiplying this mass by the acceleration of gravity, 9.8 m/s^2, gives our variable force function

$$F(x) = (9.8)(0.066)(60 - x) = 0.6468(60 - x).$$

Thus the total work performed in pulling up the rope is

$$W = \int_0^{60} 0.6468(60 - x)\, dx = 1,164.24 \text{ J}.$$

By comparison, consider the work done in lifting the entire rope 60 meters. The rope weighs $60 \times 0.066 \times 9.8 = 38.808$ N, so the work applying this force for 60 meters is $60 \times 38.808 = 2,328.48$ J. This is exactly twice the work calculated before (and we leave it to the reader to understand why.)

Example 7.5.2 Computing work performed: applying variable force

Consider again pulling a 60 m rope up a cliff face, where the rope has a mass of 66 g/m. At what point is exactly half the work performed?

Solution From Example 7.5.1 we know the total work performed is $1,164.24$ J. We want to find a height h such that the work in pulling the rope from a height of $x = 0$ to a height of $x = h$ is 582.12, half the total work. Thus we want to solve the equation

$$\int_0^{h} 0.6468(60 - x)\, dx = 582.12$$

for h.

$$\begin{aligned}\int_0^{h} 0.6468(60 - x)\, dx &= 582.12\\ \left(38.808x - 0.3234x^2\right)\Big|_0^h &= 582.12\\ 38.808h - 0.3234h^2 &= 582.12\\ -0.3234h^2 + 38.808h - 582.12 &= 0.\end{aligned}$$

Apply the Quadratic Formula.

$$h = 17.57 \text{ and } 102.43$$

As the rope is only 60 m long, the only sensible answer is $h = 17.57$. Thus about half the work is done pulling up the first 17.5 m the other half of the work is

Note: In Example 7.5.2, we find that half of the work performed in pulling up a 60 m rope is done in the last 42.43 m. Why is it not coincidental that $60/\sqrt{2} = 42.43$?

done pulling up the remaining 42.43 m.

Example 7.5.3 Computing work performed: applying variable force

A box of 100 lb of sand is being pulled up at a uniform rate a distance of 50 ft over 1 minute. The sand is leaking from the box at a rate of 1 lb/s. The box itself weighs 5 lb and is pulled by a rope weighing .2 lb/ft.

1. How much work is done lifting just the rope?
2. How much work is done lifting just the box and sand?
3. What is the total amount of work performed?

SOLUTION

1. We start by forming the force function $F_r(x)$ for the rope (where the subscript denotes we are considering the rope). As in the previous example, let x denote the amount of rope, in feet, pulled in. (This is the same as saying x denotes the height of the box.) The weight of the rope with x feet pulled in is $F_r(x) = 0.2(50-x) = 10-0.2x$. (Note that we do not have to include the acceleration of gravity here, for the *weight* of the rope per foot is given, not its *mass* per metre as before.) The work performed lifting the rope is

$$W_r = \int_0^{50} (10-0.2x)\,dx = 250 \text{ ft–lb}.$$

2. The sand is leaving the box at a rate of 1 lb/s. As the vertical trip is to take one minute, we know that 60 lb will have left when the box reaches its final height of 50 ft. Again letting x represent the height of the box, we have two points on the line that describes the weight of the sand: when $x=0$, the sand weight is 100 lb, producing the point $(0,100)$; when $x=50$, the sand in the box weighs 40 lb, producing the point $(50,40)$. The slope of this line is $\frac{100-40}{0-50} = -1.2$, giving the equation of the weight of the sand at height x as $w(x) = -1.2x+100$. The box itself weighs a constant 5 lb, so the total force function is $F_b(x) = -1.2x+105$. Integrating from $x=0$ to $x=50$ gives the work performed in lifting box and sand:

$$W_b = \int_0^{50} (-1.2x+105)\,dx = 3750 \text{ ft–lb}.$$

3. The total work is the sum of W_r and W_b: $250+3750 = 4000$ ft–lb. We can also arrive at this via integration:

$$\begin{aligned} W &= \int_0^{50} (F_r(x)+F_b(x))\,dx \\ &= \int_0^{50} (10-0.2x-1.2x+105)\,dx \\ &= \int_0^{50} (-1.4x+115)\,dx \\ &= 4000 \text{ ft–lb}. \end{aligned}$$

Hooke's Law and Springs

Hooke's Law states that the force required to compress or stretch a spring x units from its natural length is proportional to x; that is, this force is $F(x) = kx$ for some constant k. For example, if a force of 1 N stretches a given spring 2 cm, then a force of 5 N will stretch the spring 10 cm. Converting the distances to meters, we have that stretching this spring 0.02 m requires a force of $F(0.02) = k(0.02) = 1$ N, hence $k = 1/0.02 = 50$ N/m.

Fluid	lb/ft^3	kg/m^3
Concrete	150	2400
Fuel Oil	55.46	890.13
Gasoline	45.93	737.22
Iodine	307	4927
Methanol	49.3	791.3
Mercury	844	13546
Milk	63.6–65.4	1020 – 1050
Water	62.4	1000

Figure 7.5.2: Weight and Mass densities

Example 7.5.4 Computing work performed: stretching a spring

A force of 20 lb stretches a spring from a natural length of 7 inches to a length of 12 inches. How much work was performed in stretching the spring to this length?

Solution In many ways, we are not at all concerned with the actual length of the spring, only with the amount of its change. Hence, we do not care that 20 lb of force stretches the spring to a length of 12 inches, but rather that a force of 20 lb stretches the spring by 5 in. This is illustrated in Figure 7.5.1; we only measure the change in the spring's length, not the overall length of the spring.

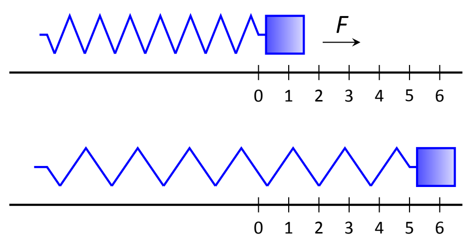


Figure 7.5.1: Illustrating the important aspects of stretching a spring in computing work in Example 7.5.4.

Converting the units of length to feet, we have

$$F(5/12) = 5/12k = 20 \text{ lb}.$$

Thus $k = 48$ lb/ft and $F(x) = 48x$.

We compute the total work performed by integrating $F(x)$ from $x = 0$ to $x = 5/12$:

$$\begin{aligned} W &= \int_0^{5/12} 48x\, dx \\ &= 24x^2\Big|_0^{5/12} \\ &= 25/6 \approx 4.1667 \text{ ft–lb}. \end{aligned}$$

Pumping Fluids

Another useful example of the application of integration to compute work comes in the pumping of fluids, often illustrated in the context of emptying a storage tank by pumping the fluid out the top. This situation is different than our previous examples for the forces involved are constant. After all, the force required to move one cubic foot of water (about 62.4 lb) is the same regardless of its location in the tank. What is variable is the distance that cubic foot of water has to travel; water closer to the top travels less distance than water at the bottom, producing less work.

We demonstrate how to compute the total work done in pumping a fluid out of the top of a tank in the next two examples.

Example 7.5.5 Computing work performed: pumping fluids

A cylindrical storage tank with a radius of 10 ft and a height of 30 ft is filled with water, which weighs approximately 62.4 lb/ft^3. Compute the amount of work performed by pumping the water up to a point 5 feet above the top of the tank.

SOLUTION We will refer often to Figure 7.5.3 which illustrates the salient aspects of this problem.

We start as we often do: we partition an interval into subintervals. We orient our tank vertically since this makes intuitive sense with the base of the tank at $y = 0$. Hence the top of the water is at $y = 30$, meaning we are interested in subdividing the y-interval $[0, 30]$ into n subintervals as

$$0 = y_1 < y_2 < \cdots < y_{n+1} = 30.$$

Consider the work W_i of pumping only the water residing in the i^{th} subinterval, illustrated in Figure 7.5.3. The force required to move this water is equal to its weight which we calculate as volume $\times$ density. The volume of water in this subinterval is $V_i = 10^2\pi\Delta y_i$; its density is 62.4 lb/ft^3. Thus the required force is $6240\pi\Delta y_i$ lb.

We approximate the distance the force is applied by using any y-value contained in the i^{th} subinterval; for simplicity, we arbitrarily use y_i for now (it will not matter later on). The water will be pumped to a point 5 feet above the top of the tank, that is, to the height of $y = 35$ ft. Thus the distance the water at height y_i travels is $35 - y_i$ ft.

In all, the approximate work W_i performed in moving the water in the i^{th} subinterval to a point 5 feet above the tank is

$$W_i \approx 6240\pi\Delta y_i(35 - y_i).$$

To approximate the total work performed in pumping out all the water from the tank, we sum all the work W_i performed in pumping the water from each of the n subintervals of $[0, 30]$:

$$W \approx \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} 6240\pi\Delta y_i(35 - y_i).$$

This is a Riemann sum. Taking the limit as the subinterval length goes to 0 gives

$$\begin{aligned} W &= \int_0^{30} 6240\pi(35 - y)\, dy \\ &= (6240\pi\left(35y - 1/2y^2\right)\Big|_0^{30} \\ &= 11,762,123 \text{ ft–lb} \\ &\approx 1.176 \times 10^7 \text{ ft–lb}. \end{aligned}$$

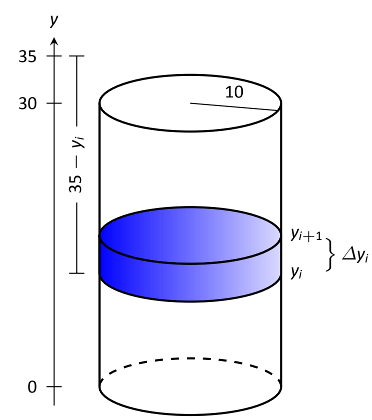


Figure 7.5.3: Illustrating a water tank in order to compute the work required to empty it in Example 7.5.5.

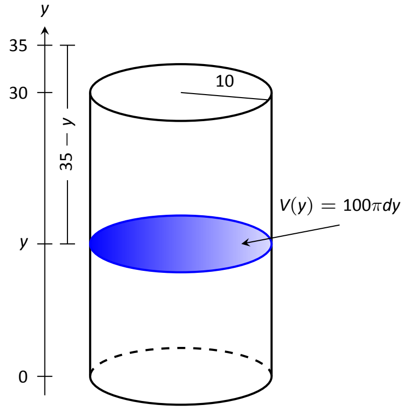


Figure 7.5.4: A simplified illustration for computing work.

We can "streamline" the above process a bit as we may now recognize what the important features of the problem are. Figure 7.5.4 shows the tank from Example 7.5.5 without the i^{th} subinterval identified. Instead, we just draw one differential element. This helps establish the height a small amount of water must travel along with the force required to move it (where the force is volume $\times$ density).

We demonstrate the concepts again in the next examples.

Example 7.5.6 Computing work performed: pumping fluids

A conical water tank has its top at ground level and its base 10 feet below ground. The radius of the cone at ground level is 2 ft. It is filled with water weighing 62.4 lb/ft^3 and is to be emptied by pumping the water to a spigot 3 feet above ground level. Find the total amount of work performed in emptying the tank.

Solution The conical tank is sketched in Figure 7.5.5. We can orient the tank in a variety of ways; we could let $y = 0$ represent the base of the tank and $y = 10$ represent the top of the tank, but we choose to keep the convention of the wording given in the problem and let $y = 0$ represent ground level and hence $y = -10$ represents the bottom of the tank. The actual "height" of the water does not matter; rather, we are concerned with the distance the water travels.

The figure also sketches a differential element, a cross–sectional circle. The radius of this circle is variable, depending on y. When $y = -10$, the circle has radius 0; when $y = 0$, the circle has radius 2. These two points, $(-10, 0)$ and $(0, 2)$, allow us to find the equation of the line that gives the radius of the cross–sectional circle, which is $r(y) = 1/5y + 2$. Hence the volume of water at this height is $V(y) = \pi(1/5y + 2)^2dy$, where dy represents a very small height of the differential element. The force required to move the water at height y is $F(y) = 62.4 \times V(y)$.

The distance the water at height y travels is given by $h(y) = 3 - y$. Thus the total work done in pumping the water from the tank is

$$\begin{aligned} W &= \int_{-10}^{0} 62.4\pi(1/5y+2)^2(3-y)\,dy \\ &= 62.4\pi \int_{-10}^{0} \left(-\frac{1}{25}y^3 - \frac{17}{25}y^2 - \frac{8}{5}y + 12\right) dy \\ &= 62.2\pi \cdot \frac{220}{3} \approx 14,376 \text{ ft–lb.} \end{aligned}$$

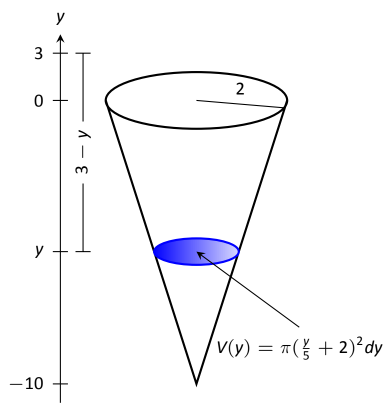


Figure 7.5.5: A graph of the conical water tank in Example 7.5.6.

Example 7.5.7 Computing work performed: pumping fluids

A rectangular swimming pool is 20 ft wide and has a 3 ft "shallow end" and a 6 ft "deep end." It is to have its water pumped out to a point 2 ft above the current top of the water. The cross–sectional dimensions of the water in the pool are given in Figure 7.5.6; note that the dimensions are for the water, not the pool itself. Compute the amount of work performed in draining the pool.

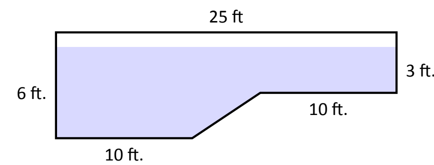


Figure 7.5.6: The cross–section of a swimming pool filled with water in Example 7.5.7.

SOLUTION For the purposes of this problem we choose to set $y = 0$ to represent the bottom of the pool, meaning the top of the water is at $y = 6$. Figure 7.5.7 shows the pool oriented with this y-axis, along with 2 differential elements as the pool must be split into two different regions.

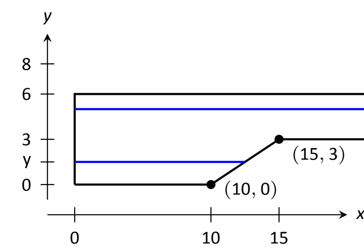


Figure 7.5.7: Orienting the pool and showing differential elements for Example 7.5.7.

The top region lies in the y-interval of $[3, 6]$, where the length of the differential element is 25 ft as shown. As the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) = 20 \cdot 25 \cdot dy$. The water is to be pumped to a height of $y = 8$, so the height function is $h(y) = 8 - y$. The work done in pumping this top region of water is

$$W_t = 62.4 \int_3^6 500(8 - y)\, dy = 327,600 \text{ ft–lb}.$$

The bottom region lies in the y-interval of $[0, 3]$; we need to compute the length of the differential element in this interval.

One end of the differential element is at $x = 0$ and the other is along the line segment joining the points $(10, 0)$ and $(15, 3)$. The equation of this line is $y = 3/5(x - 10)$; as we will be integrating with respect to y, we rewrite this equation as $x = 5/3y + 10$. So the length of the differential element is a difference of x-values: $x = 0$ and $x = 5/3y + 10$, giving a length of $x = 5/3y + 10$.

Again, as the pool is 20 ft wide, this differential element represents a thin slice of water with volume $V(y) = 20 \cdot (5/3y + 10) \cdot dy$; the height function is the same as before at $h(y) = 8 - y$. The work performed in emptying this part of the pool is

$$W_b = 62.4 \int_0^3 20(5/3y + 10)(8 - y)\, dy = 299,520 \text{ ft–lb}.$$

The total work in emptying the pool is

$$W = W_b + W_t = 327,600 + 299,520 = 627,120 \text{ ft–lb}.$$

Notice how the emptying of the bottom of the pool performs almost as much work as emptying the top. The top portion travels a shorter distance but has more water. In the end, this extra water produces more work.

The next section introduces one final application of the definite integral, the calculation of fluid force on a plate.

Exercises 7.5

Terms and Concepts

1. What are the typical units of work?

2. If a man has a mass of 80 kg on Earth, will his mass on the moon be bigger, smaller, or the same?

3. If a woman weighs 130 lb on Earth, will her weight on the moon be bigger, smaller, or the same?

4. Fill in the blanks:

   Some integrals in this section are set up by multiplying a variable ________ by a constant distance; others are set up by multiplying a constant force by a variable ________.

Problems

5. A 100 ft rope, weighing 0.1 lb/ft, hangs over the edge of a tall building.

   (a) How much work is done pulling the entire rope to the top of the building?

   (b) How much rope is pulled in when half of the total work is done?

6. A 50 m rope, with a mass density of 0.2 kg/m, hangs over the edge of a tall building.

   (a) How much work is done pulling the entire rope to the top of the building?

   (b) How much work is done pulling in the first 20 m?

7. A rope of length ℓ ft hangs over the edge of tall cliff. (Assume the cliff is taller than the length of the rope.) The rope has a weight density of d lb/ft.

   (a) How much work is done pulling the entire rope to the top of the cliff?

   (b) What percentage of the total work is done pulling in the first half of the rope?

   (c) How much rope is pulled in when half of the total work is done?

8. A 20 m rope with mass density of 0.5 kg/m hangs over the edge of a 10 m building. How much work is done pulling the rope to the top?

9. A crane lifts a 2,000 lb load vertically 30 ft with a 1” cable weighing 1.68 lb/ft.

   (a) How much work is done lifting the cable alone?

   (b) How much work is done lifting the load alone?

   (c) Could one conclude that the work done lifting the cable is negligible compared to the work done lifting the load?

10. A 100 lb bag of sand is lifted uniformly 120 ft in one minute. Sand leaks from the bag at a rate of 1/4 lb/s. What is the total work done in lifting the bag?

11. A box weighing 2 lb lifts 10 lb of sand vertically 50 ft. A crack in the box allows the sand to leak out such that 9 lb of sand is in the box at the end of the trip. Assume the sand leaked out at a uniform rate. What is the total work done in lifting the box and sand?

12. A force of 1000 lb compresses a spring 3 in. How much work is performed in compressing the spring?

13. A force of 2 N stretches a spring 5 cm. How much work is performed in stretching the spring?

14. A force of 50 lb compresses a spring from a natural length of 18 in to 12 in. How much work is performed in compressing the spring?

15. A force of 20 lb stretches a spring from a natural length of 6 in to 8 in. How much work is performed in stretching the spring?

16. A force of 7 N stretches a spring from a natural length of 11 cm to 21 cm. How much work is performed in stretching the spring from a length of 16 cm to 21 cm?

17. A force of f N stretches a spring d m from its natural length. How much work is performed in stretching the spring?

18. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.

    How much work is done in lifting the box 1.5 ft (i.e, the spring will be stretched 1 ft beyond its natural length)?

19. A 20 lb weight is attached to a spring. The weight rests on the spring, compressing the spring from a natural length of 1 ft to 6 in.

    How much work is done in lifting the box 6 in (i.e, bringing the spring back to its natural length)?

20. A 5 m tall cylindrical tank with radius of 2 m is filled with 3 m of gasoline, with a mass density of 737.22 kg/m^3. Compute the total work performed in pumping all the gasoline to the top of the tank.

21. A 6 ft cylindrical tank with a radius of 3 ft is filled with water, which has a weight density of 62.4 lb/ft^3. The water is to be pumped to a point 2 ft above the top of the tank.

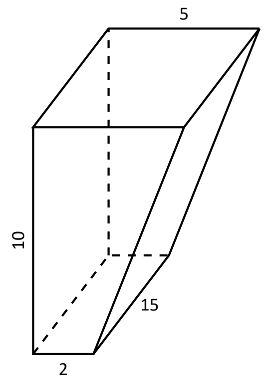
    (a) How much work is performed in pumping all the water from the tank?

    (b) How much work is performed in pumping 3 ft of water from the tank?

    (c) At what point is 1/2 of the total work done?

22. A gasoline tanker is filled with gasoline with a weight density of 45.93 lb/ft^3. The dispensing valve at the base is jammed shut, forcing the operator to empty the tank via pumping the gas to a point 1 ft above the top of the tank. Assume the tank is a perfect cylinder, 20 ft long with a diameter of 7.5 ft. How much work is performed in pumping all the gasoline from the tank?

23. A fuel oil storage tank is 10 ft deep with trapezoidal sides, 5 ft at the top and 2 ft at the bottom, and is 15 ft wide (see diagram below). Given that fuel oil weighs 55.46 lb/ft^3, find the work performed in pumping all the oil from the tank to a point 3 ft above the top of the tank.



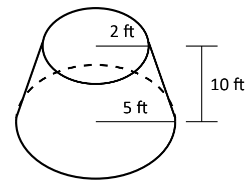
24. A conical water tank is 5 m deep with a top radius of 3 m. (This is similar to Example 7.5.6.) The tank is filled with pure water, with a mass density of 1000 kg/m^3.

    (a) Find the work performed in pumping all the water to the top of the tank.

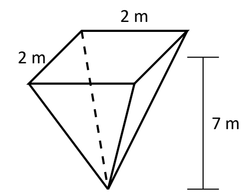
    (b) Find the work performed in pumping the top 2.5 m of water to the top of the tank.

    (c) Find the work performed in pumping the top half of the water, by volume, to the top of the tank.

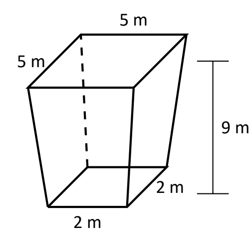
25. A water tank has the shape of a truncated cone, with dimensions given below, and is filled with water with a weight density of 62.4 lb/ft^3. Find the work performed in pumping all water to a point 1 ft above the top of the tank.



26. A water tank has the shape of an inverted pyramid, with dimensions given below, and is filled with water with a mass density of 1000 kg/m^3. Find the work performed in pumping all water to a point 5 m above the top of the tank.



27. A water tank has the shape of an truncated, inverted pyramid, with dimensions given below, and is filled with water with a mass density of 1000 kg/m^3. Find the work performed in pumping all water to a point 1 m above the top of the tank.



7.6 Fluid Forces

In the unfortunate situation of a car driving into a body of water, the conventional wisdom is that the water pressure on the doors will quickly be so great that they will be effectively unopenable. (Survival techniques suggest immediately opening the door, rolling down or breaking the window, or waiting until the water fills up the interior at which point the pressure is equalized and the door will open. See Mythbusters episode #72 to watch Adam Savage test these options.)

How can this be true? How much force does it take to open the door of a submerged car? In this section we will find the answer to this question by examining the forces exerted by fluids.

We start with **pressure**, which is related to **force** by the following equations:

$$\text{Pressure} = \frac{\text{Force}}{\text{Area}} \quad \Leftrightarrow \quad \text{Force} = \text{Pressure} \times \text{Area}.$$

In the context of fluids, we have the following definition.

Definition 7.6.1 Fluid Pressure

Let w be the weight–density of a fluid. The **pressure** p exerted on an object at depth d in the fluid is $p = w \cdot d$.

We use this definition to find the **force** exerted on a horizontal sheet by considering the sheet's area.

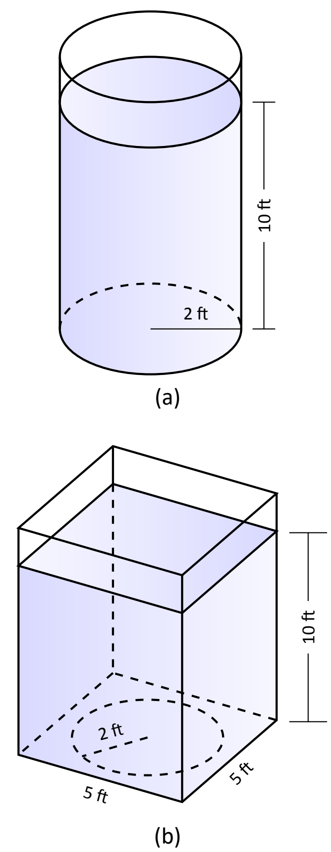


Figure 7.6.1: The cylindrical and rectangular tank in Example 7.6.1.

Example 7.6.1 Computing fluid force

1. A cylindrical storage tank has a radius of 2 ft and holds 10 ft of a fluid with a weight–density of 50 lb/ft^3. (See Figure 7.6.1(a).) What is the force exerted on the base of the cylinder by the fluid?
2. A rectangular tank whose base is a 5 ft square has a circular hatch at the bottom with a radius of 2 ft. The tank holds 10 ft of a fluid with a weight–density of 50 lb/ft^3. (See Figure 7.6.1(b).) What is the force exerted on the hatch by the fluid?

Solution

1. Using Definition 7.6.1, we calculate that the pressure exerted on the cylinder's base is $w \cdot d = 50\text{ lb/ft}^3 \times 10\text{ ft} = 500\text{ lb/ft}^2$. The area of the base is $\pi \cdot 2^2 = 4\pi\text{ ft}^2$. So the force exerted by the fluid is
$$F = 500 \times 4\pi = 6283\text{ lb}.$$
Note that we effectively just computed the *weight* of the fluid in the tank.

2. The dimensions of the tank in this problem are irrelevant. All we are concerned with are the dimensions of the hatch and the depth of the fluid. Since the dimensions of the hatch are the same as the base of the tank in the previous part of this example, as is the depth, we see that the fluid force is the same. That is, $F = 6283$ lb.

   A key concept to understand here is that we are effectively measuring the weight of a 10 ft column of water above the hatch. The size of the tank holding the fluid does not matter.

The previous example demonstrates that computing the force exerted on a horizontally oriented plate is relatively easy to compute. What about a vertically oriented plate? For instance, suppose we have a circular porthole located on the side of a submarine. How do we compute the fluid force exerted on it?

Pascal's Principle states that the pressure exerted by a fluid at a depth is equal in all directions. Thus the pressure on any portion of a plate that is 1 ft below the surface of water is the same no matter how the plate is oriented. (Thus a hollow cube submerged at a great depth will not simply be "crushed" from above, but the sides will also crumple in. The fluid will exert force on *all* sides of the cube.)

So consider a vertically oriented plate as shown in Figure 7.6.2 submerged in a fluid with weight–density w. What is the total fluid force exerted on this plate? We find this force by first approximating the force on small horizontal strips.

Let the top of the plate be at depth b and let the bottom be at depth a. (For now we assume that surface of the fluid is at depth 0, so if the bottom of the plate is 3 ft under the surface, we have $a = -3$. We will come back to this later.) We partition the interval $[a, b]$ into n subintervals

$$a = y_1 < y_2 < \cdots < y_{n+1} = b,$$

with the i^{th} subinterval having length Δy_i. The force F_i exerted on the plate in the i^{th} subinterval is $F_i = \text{Pressure} \times \text{Area}$.

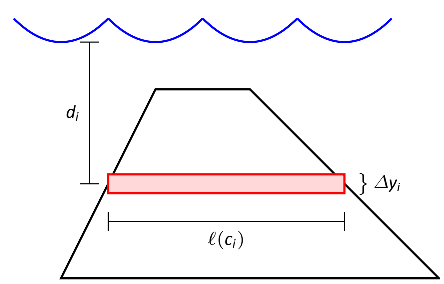


Figure 7.6.2: A thin, vertically oriented plate submerged in a fluid with weight–density w.

The pressure is depth $\times w$. We approximate the depth of this thin strip by choosing any value d_i in $[y_i, y_{i+1}]$; the depth is approximately $-d_i$. (Our convention has d_i being a negative number, so $-d_i$ is positive.) For convenience, we let d_i be an endpoint of the subinterval; we let $d_i = y_i$.

The area of the thin strip is approximately length $\times$ width. The width is Δy_i. The length is a function of some y-value c_i in the i^{th} subinterval. We state the length is $\ell(c_i)$. Thus

$$\begin{aligned} F_i &= \text{Pressure} \times \text{Area} \\ &= -y_i \cdot w \times \ell(c_i) \cdot \Delta y_i. \end{aligned}$$

To approximate the total force, we add up the approximate forces on each of the n thin strips:

$$F = \sum_{i=1}^{n} F_i \approx \sum_{i=1}^{n} -w \cdot y_i \cdot \ell(c_i) \cdot \Delta y_i.$$

This is, of course, another Riemann Sum. We can find the exact force by taking a limit as the subinterval lengths go to 0; we evaluate this limit with a definite integral.

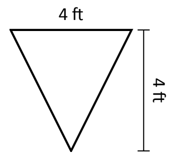


Figure 7.6.3: A thin plate in the shape of an isosceles triangle in Example 7.6.2.

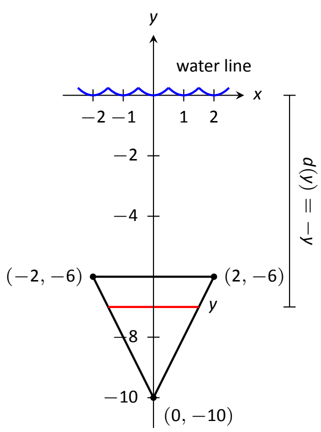


Figure 7.6.4: Sketching the triangular plate in Example 7.6.2 with the convention that the water level is at $y = 0$.

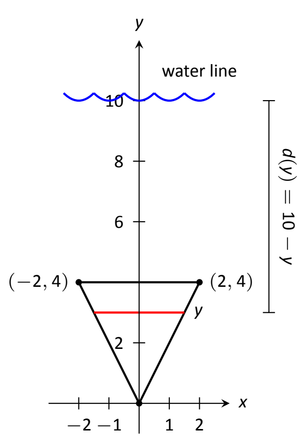


Figure 7.6.5: Sketching the triangular plate in Example 7.6.2 with the convention that the base of the triangle is at $(0, 0)$.

Key Idea 7.6.1 Fluid Force on a Vertically Oriented Plate

Let a vertically oriented plate be submerged in a fluid with weight–density w where the top of the plate is at $y = b$ and the bottom is at $y = a$. Let $\ell(y)$ be the length of the plate at y.

1. If $y = 0$ corresponds to the surface of the fluid, then the force exerted on the plate by the fluid is
$$F = \int_a^b w \cdot (-y) \cdot \ell(y)\, dy.$$
2. In general, let $d(y)$ represent the distance between the surface of the fluid and the plate at y. Then the force exerted on the plate by the fluid is
$$F = \int_a^b w \cdot d(y) \cdot \ell(y)\, dy.$$

Example 7.6.2 Finding fluid force

Consider a thin plate in the shape of an isosceles triangle as shown in Figure 7.6.3 submerged in water with a weight–density of 62.4 lb/ft^3. If the bottom of the plate is 10 ft below the surface of the water, what is the total fluid force exerted on this plate?

Solution We approach this problem in two different ways to illustrate the different ways Key Idea 7.6.1 can be implemented. First we will let $y = 0$ represent the surface of the water, then we will consider an alternate convention.

1. We let $y = 0$ represent the surface of the water; therefore the bottom of the plate is at $y = -10$. We center the triangle on the y-axis as shown in Figure 7.6.4. The depth of the plate at y is $-y$ as indicated by the Key Idea. We now consider the length of the plate at y.

   We need to find equations of the left and right edges of the plate. The right hand side is a line that connects the points $(0, -10)$ and $(2, -6)$: that line has equation $x = 1/2(y + 10)$. (Find the equation in the familiar $y = mx+b$ format and solve for x.) Likewise, the left hand side is described by the line $x = -1/2(y + 10)$. The total length is the distance between these two lines: $\ell(y) = 1/2(y + 10) - (-1/2(y + 10)) = y + 10$.

   The total fluid force is then:
$$\begin{aligned} F &= \int_{-10}^{-6} 62.4(-y)(y + 10)\, dy \\ &= 62.4 \cdot \frac{176}{3} \approx 3660.8 \text{ lb.} \end{aligned}$$

2. Sometimes it seems easier to orient the thin plate nearer the origin. For instance, consider the convention that the bottom of the triangular plate is at $(0, 0)$, as shown in Figure 7.6.5. The equations of the left and right hand sides are easy to find. They are $y = 2x$ and $y = -2x$, respectively, which we rewrite as $x = 1/2y$ and $x = -1/2y$. Thus the length function is $\ell(y) = 1/2y - (-1/2y) = y$.

   As the surface of the water is 10 ft above the base of the plate, we have that the surface of the water is at $y = 10$. Thus the depth function is the distance between $y = 10$ and y; $d(y) = 10 - y$. We compute the total fluid force as:

$$F = \int_0^4 62.4(10-y)(y)\,dy$$
$$\approx 3660.8 \text{ lb}.$$

The correct answer is, of course, independent of the placement of the plate in the coordinate plane as long as we are consistent.

Example 7.6.3 Finding fluid force

Find the total fluid force on a car door submerged up to the bottom of its window in water, where the car door is a rectangle 40" long and 27" high (based on the dimensions of a 2005 Fiat Grande Punto.)

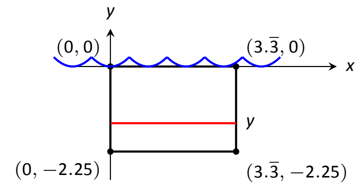


Figure 7.6.6: Sketching a submerged car door in Example 7.6.3.

Solution The car door, as a rectangle, is drawn in Figure 7.6.6. Its length is $10/3$ ft and its height is 2.25 ft. We adopt the convention that the top of the door is at the surface of the water, both of which are at $y = 0$. Using the weight–density of water of 62.4 lb/ft^3, we have the total force as

$$\begin{aligned} F &= \int_{-2.25}^{0} 62.4(-y)10/3\,dy \\ &= \int_{-2.25}^{0} -208y\,dy \\ &= -104y^2\Big|_{-2.25}^{0} \\ &= 526.5 \text{ lb}. \end{aligned}$$

Most adults would find it very difficult to apply over 500 lb of force to a car door while seated inside, making the door effectively impossible to open. This is counter–intuitive as most assume that the door would be relatively easy to open. The truth is that it is not, hence the survival tips mentioned at the beginning of this section.

Example 7.6.4 Finding fluid force

An underwater observation tower is being built with circular viewing portholes enabling visitors to see underwater life. Each vertically oriented porthole is to have a 3 ft diameter whose center is to be located 50 ft underwater. Find the total fluid force exerted on each porthole. Also, compute the fluid force on a horizontally oriented porthole that is under 50 ft of water.

SOLUTION We place the center of the porthole at the origin, meaning the surface of the water is at $y = 50$ and the depth function will be $d(y) = 50 - y$; see Figure 7.6.7

The equation of a circle with a radius of 1.5 is $x^2 + y^2 = 2.25$; solving for x we have $x = \pm\sqrt{2.25 - y^2}$, where the positive square root corresponds to the right side of the circle and the negative square root corresponds to the left side of the circle. Thus the length function at depth y is $\ell(y) = 2\sqrt{2.25 - y^2}$. Integrating on $[-1.5, 1.5]$ we have:

$$\begin{aligned} F &= 62.4\int_{-1.5}^{1.5} 2(50-y)\sqrt{2.25-y^2}\,dy \\ &= 62.4\int_{-1.5}^{1.5} \left(100\sqrt{2.25-y^2} - 2y\sqrt{2.25-y^2}\right)dy \\ &= 6240\int_{-1.5}^{1.5} \left(\sqrt{2.25-y^2}\right)dy - 62.4\int_{-1.5}^{1.5}\left(2y\sqrt{2.25-y^2}\right)dy. \end{aligned}$$

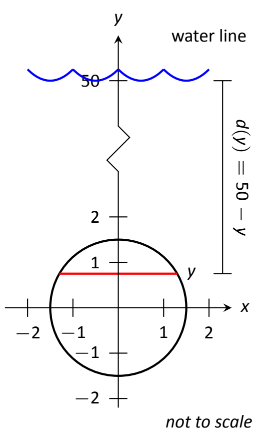


Figure 7.6.7: Measuring the fluid force on an underwater porthole in Example 7.6.4.

The second integral above can be evaluated using substitution. Let $u = 2.25 - y^2$ with $du = -2y\,dy$. The new bounds are: $u(-1.5) = 0$ and $u(1.5) = 0$; the new integral will integrate from $u = 0$ to $u = 0$, hence the integral is 0.

The first integral above finds the area of half a circle of radius 1.5, thus the first integral evaluates to $6240 \cdot \pi \cdot 1.5^2/2 = 22,054$. Thus the total fluid force on a vertically oriented porthole is 22,054 lb.

Finding the force on a horizontally oriented porthole is more straightforward:

$$F = \text{Pressure} \times \text{Area} = 62.4 \cdot 50 \times \pi \cdot 1.5^2 = 22,054 \text{ lb}.$$

That these two forces are equal is not coincidental; it turns out that the fluid force applied to a vertically oriented circle whose center is at depth d is the same as force applied to a horizontally oriented circle at depth d.

We end this chapter with a reminder of the true skills meant to be developed here. We are not truly concerned with an ability to find fluid forces or the volumes of solids of revolution. Work done by a variable force is important, though measuring the work done in pulling a rope up a cliff is probably not.

What we are actually concerned with is the ability to solve certain problems by first approximating the solution, then refining the approximation, then recognizing if/when this refining process results in a definite integral through a limit. Knowing the formulas found inside the special boxes within this chapter is beneficial as it helps solve problems found in the exercises, and other mathematical skills are strengthened by properly applying these formulas. However, more importantly, understand how each of these formulas was constructed. Each is the result of a summation of approximations; each summation was a Riemann sum, allowing us to take a limit and find the exact answer through a definite integral.

The next chapter addresses an entirely different topic: sequences and series. In short, a sequence is a list of numbers, where a series is the summation of a list of numbers. These seemingly–simple ideas lead to very powerful mathematics.

Exercises 7.6

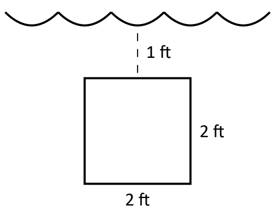
Terms and Concepts

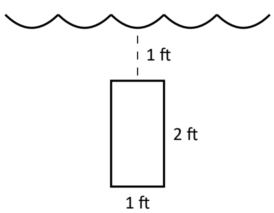
1. State in your own words Pascal's Principle.

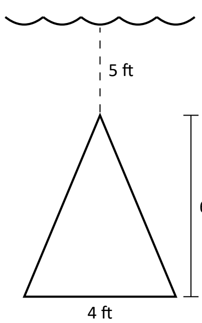
2. State in your own words how pressure is different from force.

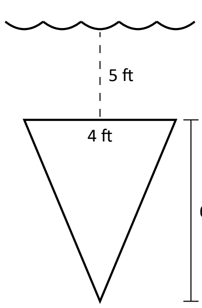
Problems

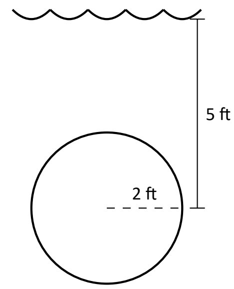
In Exercises 3 – 12, find the fluid force exerted on the given plate, submerged in water with a weight density of 62.4 lb/ft^3.

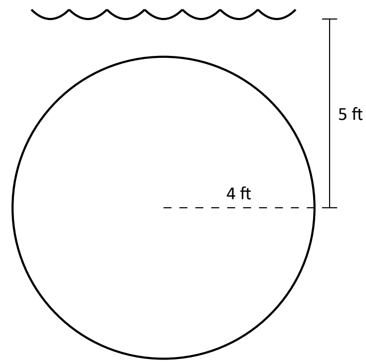
3. 

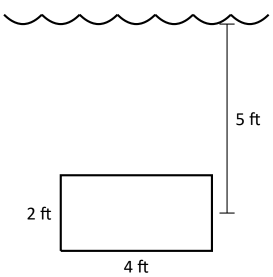
4. 

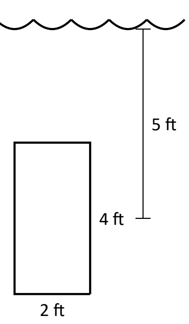
5. 

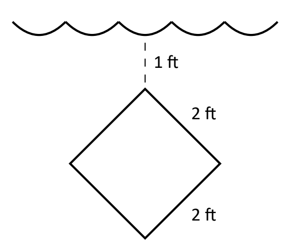
6. 

7. 

8. 

9. 

10. 

11. 

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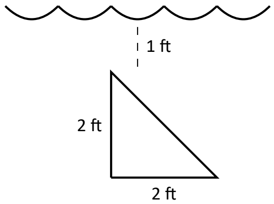
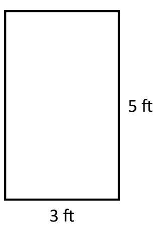


In Exercises 13 – 18, the side of a container is pictured. Find the fluid force exerted on this plate when the container is full of:

1. **water, with a weight density of 62.4 lb/ft^3, and**
2. **concrete, with a weight density of 150 lb/ft^3.**

13.



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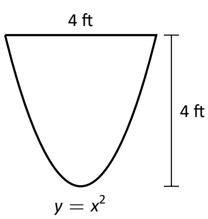

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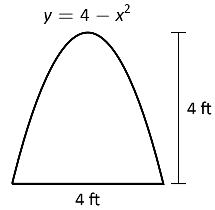

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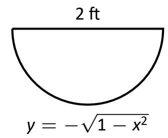

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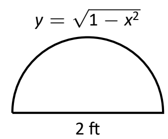

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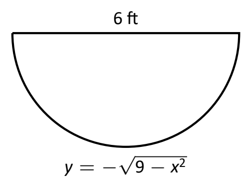

19. How deep must the center of a vertically oriented circular plate with a radius of 1 ft be submerged in water, with a weight density of 62.4 lb/ft^3, for the fluid force on the plate to reach 1,000 lb?

20. How deep must the center of a vertically oriented square plate with a side length of 2 ft be submerged in water, with a weight density of 62.4 lb/ft^3, for the fluid force on the plate to reach 1,000 lb?

8: Differential Equations

8.1 Introduction to differential equations

Differential equations

The laws of physics are generally written down as differential equations. Therefore, all of science and engineering use differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science and engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned. As an analogy, suppose all your classes from now on were given in Swahili. It would be important to first learn Swahili, or you would have a very tough time getting a good grade in your classes.

You have already seen many differential equations without perhaps knowing about it. And you have even solved simple differential equations when you were taking calculus. Let us see an example you may not have seen:

$$\frac{dx}{dt} + x = 2\cos t. \tag{8.1}$$

Here x is the **dependent variable** and t is the **independent variable**. Equation (8.1) is a basic example of a **differential equation**. In fact it is an example of a **first order differential equation**, since it involves only the first derivative of the dependent variable. This equation arises from Newton's law of cooling where the ambient temperature oscillates with time.

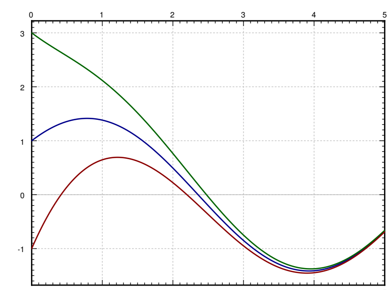

Figure 8.1.1: A few solutions of $\frac{dx}{dt} + x = 2\cos t$.

Solutions of differential equations

Solving the differential equation means finding x in terms of t. That is, we want to find a function of t, which we will call x, such that when we plug x, t, and $\frac{dx}{dt}$ into (8.1), the equation holds. It is the same idea as it would be for a normal (algebraic) equation of just x and t. We claim that

$$x = x(t) = \cos t + \sin t$$

is a **solution**. How do we check? We simply plug x into equation (8.1)! First we need to compute $\frac{dx}{dt}$. We find that $\frac{dx}{dt} = -\sin t + \cos t$. Now let us compute the left hand side of (8.1).

$$\frac{dx}{dt} + x = (-\sin t + \cos t) + (\cos t + \sin t) = 2\cos t.$$

Hooray! We got precisely the right hand side. But there is more! We claim $x = \cos t + \sin t + e^{-t}$ is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}.$$

Again plugging into the left hand side of (8.1)

$$\frac{dx}{dt} + x = (-\sin t + \cos t - e^{-t}) + (\cos t + \sin t + e^{-t}) = 2\cos t.$$

And it works yet again!

So there can be many different solutions. In fact, for this equation all solutions can be written in the form

$$x = \cos t + \sin t + Ce^{-t}$$

for some constant C. See 8.1.1 for the graph of a few of these solutions. We will see how we find these solutions a few lectures from now.

It turns out that solving differential equations can be quite hard. There is no general method that solves every differential equation. We will generally focus on how to get exact formulas for solutions of certain differential equations, but we will also spend a little bit of time on getting approximate solutions.

For most of the course we will look at **ordinary differential equation** (often abbreviated **ODEs**, by which we mean that there is only one independent variable and derivatives are only with respect to this one variable. If there are several independent variables, we will get **partial differential equations** or **PDEs**.

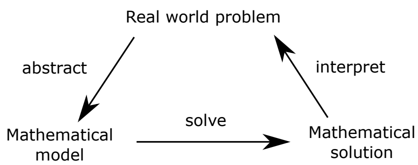

Figure 8.1.2: Mathematical modelling process

Even for ODEs, which are very well understood, it is not a simple question of turning a crank to get answers. It is important to know when it is easy to find solutions and how to do so. Although in real applications you will leave much of the actual calculations to computers, you need to understand what they are doing. It is often necessary to simplify or transform your equations into something that a computer can understand and solve. You may need to make certain assumptions and changes in your model to achieve this.

To be a successful engineer or scientist, you will be required to solve problems in your job that you have never seen before. It is important to learn problem solving techniques, so that you may apply those techniques to new problems. A common mistake is to expect to learn some prescription for solving all the problems you will encounter in your later career. This course is no exception.

Differential equations in practice

So how do we use differential equations in science and engineering? First, we have some **real world problem** we wish to understand. We make some simplifying assumptions and create a **mathematical model** That is, we translate the real world situation into a set of differential equations. Then we apply mathematics to get some sort of a **mathematical solution** to the model. There is still something left to do. We have to interpret the results. We have to figure out what the mathematical solution says about the real world problem we started with.

Learning how to formulate the mathematical model and how to interpret the results is what your physics and engineering classes do. In this course we will focus mostly on the mathematical analysis. Sometimes we will work with simple real world examples, so that we have some intuition and motivation about what we are doing.

Let us look at an example of this process. One of the most basic differential equations is the standard **exponential growth model** Let P denote the population of some bacteria on a Petri dish. We assume that there is enough food and enough space. Then the rate of growth of bacteria is proportional to the population—a large population grows quicker. Let t denote time (say in seconds) and P the population. Our model is

$$\frac{dP}{dt} = kP,$$

for some positive constant $k > 0$.

Example 8.1.1 Model for bacterial growth

Suppose there are 100 bacteria at time 0 and 200 bacteria 10 seconds later. How many bacteria will there be 1 minute from time 0 (in 60 seconds)?

Solution First we have to solve the equation. We claim that a solution is given by

$$P(t) = Ce^{kt},$$

where C is a constant. Let us try:

$$\frac{dP}{dt} = Cke^{kt} = kP.$$

And it really is a solution.

OK, so what now? We do not know C and we do not know k. But we know something. We know $P(0) = 100$, and we also know $P(10) = 200$. Let us plug these conditions in and see what happens.

$$100 = P(0) = Ce^{k0} = C,$$
$$200 = P(10) = 100\, e^{k10}.$$

Therefore, $2 = e^{10k}$ or $\frac{\ln 2}{10} = k \approx 0.069$. So we know that

$$P(t) = 100\, e^{(\ln 2)t/10} \approx 100\, e^{0.069t}.$$

At one minute, $t = 60$, the population is $P(60) = 6400$. See Figure 8.1.3.

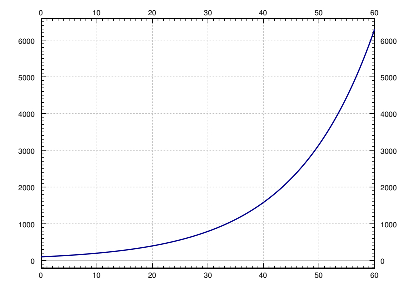

Figure 8.1.3: Bacteria growth in the first 60 seconds.

Let us talk about the interpretation of the results. Does our solution mean that there must be exactly 6400 bacteria on the plate at 60s? No! We made assumptions that might not be true exactly, just approximately. If our assumptions are reasonable, then there will be approximately 6400 bacteria. Also, in real life P is a discrete quantity, not a real number. However, our model has no problem saying that for example at 61 seconds, $P(61) \approx 6859.35$.

Normally, the k in $P' = kP$ is known, and we want to solve the equation for different **initial conditions**. What does that mean? Take $k = 1$ for simplicity. Now suppose we want to solve the equation $\frac{dP}{dt} = P$ subject to $P(0) = 1000$ (the initial condition). Then the solution turns out to be (exercise)

$$P(t) = 1000\, e^{t}.$$

We call $P(t) = Ce^{t}$ **general solution**, as every solution of the equation can be written in this form for some constant C. You will need an initial condition to find out what C is, in order to find the **particular solution** we are looking for. Generally, when we say "particular solution," we just mean some solution.

Let us get to what we will call the four fundamental equations. These equations appear very often and it is useful to just memorize what their solutions are. These solutions are reasonably easy to guess by recalling properties of exponentials, sines, and cosines. They are also simple to check, which is something that you should always do. There is no need to wonder if you have remembered the solution correctly.

First such equation is,

$$\frac{dy}{dx} = ky,$$

for some constant $k > 0$. Here y is the dependent and x the independent variable. The general solution for this equation is

$$y(x) = Ce^{kx}.$$

We have already seen that this function is a solution above with different variable names.

Next,

$$\frac{dy}{dx} = -ky,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = Ce^{-kx}.$$

Exercise: Check that the y given is really a solution to the equation.

Next, take the **second order differential equation**

$$\frac{d^2y}{dx^2} = -k^2y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx).$$

Note that because we have a second order differential equation, we have two constants in our general solution.

Exercise: Check that the y given is really a solution to the equation.

And finally, take the second order differential equation

$$\frac{d^2y}{dx^2} = k^2y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1e^{kx} + C_2e^{-kx},$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(kx).$$

For those that do not know, cosh and sinh are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2},$$
$$\sinh x = \frac{e^x - e^{-x}}{2}.$$

These functions are sometimes easier to work with than exponentials. They have some nice familiar properties such as $\cosh 0 = 1$, $\sinh 0 = 0$, and $\frac{d}{dx} \cosh x = \sinh x$ (no that is not a typo) and $\frac{d}{dx} \sinh x = \cosh x$.

Exercise: Check that both forms of the y given are really solutions to the equation.

An interesting note about cosh: The graph of cosh is the exact shape a hanging chain will make. This shape is called a **catenary** Contrary to popular belief this is not a parabola. If you invert the graph of cosh it is also the ideal arch for supporting its own weight. For example, the gateway arch in Saint Louis is an inverted graph of cosh—if it were just a parabola it might fall down. The formula used in the design is inscribed inside the arch:

$$y = -127.7 \text{ ft} \cdot \cosh(x/127.7 \text{ ft}) + 757.7 \text{ ft}.$$

Exercises 8.1

Problems

1. Show that $x = e^{4t}$ is a solution to $x''' - 12x'' + 48x' - 64x = 0$.

2. Show that $x = e^{t}$ is not a solution to $x''' - 12x'' + 48x' - 64x = 0$.

3. Is $y = \sin t$ a solution to $\left(\dfrac{dy}{dt}\right)^2 = 1 - y^2$? Justify.

4. Let $y'' + 2y' - 8y = 0$. Now try a solution of the form $y = e^{rx}$ for some (unknown) constant r. Is this a solution for some r? If so, find all such r.

5. Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.

6. Verify that $x = C_1e^{-t} + C_2e^{2t}$ is a solution to $x'' - x' - 2x = 0$. Find C_1 and C_2 to solve for the initial conditions $x(0) = 10$ and $x'(0) = 0$.

7. Find a solution to $(x')^2 + x^2 = 4$ using your knowledge of derivatives of functions that you know from basic calculus.

8. Solve:

   (a) $\dfrac{dA}{dt} = -10A,\ A(0) = 5$

   (b) $\dfrac{dH}{dx} = 3H,\ H(0) = 1$

   (c) $\dfrac{d^2y}{dx^2} = 4y,\ y(0) = 0,\ y'(0) = 1$

   (d) $\dfrac{d^2x}{dy^2} = -9x,\ x(0) = 1,\ x'(0) = 0$

9. Is there a solution to $y' = y$, such that $y(0) = y(1)$?

10. Show that $x = e^{-2t}$ is a solution to $x'' + 4x' + 4x = 0$.

11. Is $y = x^2$ a solution to $x^2y'' - 2y = 0$? Justify.

12. Let $xy'' - y' = 0$. Try a solution of the form $y = x^r$. Is this a solution for some r? If so, find all such r.

13. Verify that $x = C_1e^t + C_2$ is a solution to $x'' - x' = 0$. Find C_1 and C_2 so that x satisfies $x(0) = 10$ and $x'(0) = 100$.

14. Solve $\frac{d\varphi}{ds} = 8\varphi$ and $\varphi(0) = -9$.

8.2 Integrals as solutions

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y),$$

or just

$$y' = f(x, y).$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few lectures we will look at special cases where solutions are not difficult to obtain. In this section, let us assume that f is a function of x alone, that is, the equation is

$$y' = f(x). \tag{8.2}$$

We could just integrate (antidifferentiate) both sides with respect to x.

$$\int y'(x)\, dx = \int f(x)\, dx + C,$$

that is

$$y(x) = \int f(x)\, dx + C.$$

This $y(x)$ is actually the general solution. So to solve (8.2), we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called indefinite integral. The is really the **antiderivative** (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write $\int f(x)\, dx + C$ as

$$\int_{x_0}^{x} f(t)\, dt + C.$$

Hence the terminology *to integrate* when we may really mean *to antidifferentiate*. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should *always* think of the definite integral.

Example 8.2.1 Finding a general solution

Find the general solution of $y' = 3x^2$.

Solution Elementary calculus tells us that the general solution must be $y = x^3 + C$. Let us check by differentiating: $y' = 3x^2$. We have gotten *precisely* our equation back.

Normally, we also have an initial condition such as $y(x_0) = y_0$ for some two numbers x_0 and y_0 (x_0 is usually 0, but not always). We can then write the solution as a definite integral in a nice way. Suppose our problem is $y' = f(x)$, $y(x_0) = y_0$. Then the solution is

$$y(x) = \int_{x_0}^{x} f(s)\, ds + y_0. \tag{8.3}$$

Let us check! We compute $y' = f(x)$, via the fundamental theorem of calculus, and by Jupiter, y is a solution. Is it the one satisfying the initial condition? Well, $y(x_0) = \int_{x_0}^{x_0} f(x)\, dx + y_0 = y_0$. It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, (8.3) is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

Example 8.2.2 **An ODE with no closed-form solution**

Solve

$$y' = e^{-x^2}, \qquad y(0) = 1.$$

Solution By the preceding discussion, the solution must be

$$y(x) = \int_0^x e^{-s^2}\, ds + 1.$$

Here is a good way to make fun of your friends taking second semester calculus. Tell them to find the closed form solution. Ha ha ha (bad math joke). It is not possible (in closed form). There is absolutely nothing wrong with writing the solution as a definite integral. This particular integral is in fact very important in statistics.

Using this method, we can also solve equations of the form

$$y' = f(y).$$

Let us write the equation in .

$$\frac{dy}{dx} = f(y).$$

Now we use the inverse function theorem from calculus to switch the roles of x and y to obtain

$$\frac{dx}{dy} = \frac{1}{f(y)}.$$

What we are doing seems like algebra with dx and dy. It is tempting to just do algebra with dx and dy as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point we can simply integrate,

$$x(y) = \int \frac{1}{f(y)}\, dy + C.$$

Finally, we try to solve for y.

Example 8.2.3 **Solving the exponential growth equation**

Previously, we guessed $y' = ky$ (for some $k > 0$) has the solution $y = Ce^{kx}$. We can now find the solution without guessing.

Solution First we note that $y = 0$ is a solution. Henceforth, we assume $y \neq 0$. We write

$$\frac{dx}{dy} = \frac{1}{ky}.$$

We integrate to obtain

$$x(y) = x = \frac{1}{k} \ln |y| + D,$$

where D is an arbitrary constant. Now we solve for y (actually for $|y|$).

$$|y| = e^{kx-kD} = e^{-kD} e^{kx}.$$

If we replace e^{-kD} with an arbitrary constant C we can get rid of the absolute value bars (which we can do as D was arbitrary). In this way, we also incorporate the solution $y = 0$. We get the same general solution as we guessed before, $y = Ce^{kx}$.

Example 8.2.4 Solving an ODE by integration
Find the general solution of $y' = y^2$.

Solution First we note that $y = 0$ is a solution. We can now assume that $y \neq 0$. Write

$$\frac{dx}{dy} = \frac{1}{y^2}.$$

We integrate to get

$$x = \frac{-1}{y} + C.$$

We solve for $y = \frac{1}{C-x}$. So the general solution is

$$y = \frac{1}{C - x} \qquad \text{or} \qquad y = 0.$$

Note the singularities of the solution. If for example $C = 1$, then the solution "blows up" as we approach $x = 1$. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The equation $y' = y^2$ is very nice and defined everywhere, but the solution is only defined on some interval $(-\infty, C)$ or (C, ∞).

Classical problems leading to differential equations solvable by integration are problems dealing with , and . You have surely seen these problems before in your calculus class.

Example 8.2.5 Finding the distance travelled
Suppose a car drives at a speed $e^{t/2}$ metres per second, where t is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

Solution Let x denote the distance the car travelled. The equation is

$$x' = e^{t/2}.$$

We can just integrate this equation to get that

$$x(t) = 2e^{t/2} + C.$$

We still need to figure out C. We know that when $t = 0$, then $x = 0$. That is, $x(0) = 0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C.$$

Thus $C = -2$ and

$$x(t) = 2e^{t/2} - 2.$$

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44 \text{ metres}, \qquad x(10) = 2e^{10/2} - 2 \approx 294 \text{ metres}.$$

Example 8.2.6 Another car problem

Suppose that the car accelerates at a rate of t^2 m/s². At time $t = 0$ the car is at the 1 metre mark and is travelling at 10 m/s. Where is the car at time $t = 10$?

SOLUTION Well this is actually a second order problem. If x is the distance travelled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is

$$x'' = t^2, \qquad x(0) = 1, \qquad x'(0) = 10.$$

What if we say $x' = v$. Then we have the problem

$$v' = t^2, \qquad v(0) = 10.$$

Once we solve for v, we can integrate and find x.

Exercise: Solve for v, and then solve for x. Find $x(10)$ to answer the question.

Exercises 8.2

Problems

1. Solve $\frac{dy}{dx} = x^2 + x$ for $y(1) = 3$.

2. Solve $\frac{dy}{dx} = \sin(5x)$ for $y(0) = 2$.

3. Solve $\frac{dy}{dx} = \frac{1}{x^2 - 1}$ for $y(0) = 0$.

4. Solve $y' = y^3$ for $y(0) = 1$.

5. (A little harder) Solve $y' = (y-1)(y+1)$ for $y(0) = 3$.

6. Solve $\frac{dy}{dx} = \frac{1}{y+1}$ for $y(0) = 0$.

7. (Harder) Solve $y'' = \sin x$ for $y(0) = 0$, $y'(0) = 2$.

8. A spaceship is travelling at the speed $2t^2 + 1$ km/s (t is time in seconds). It is pointing directly away from Earth and at time $t = 0$ it is 1000 kilometres from earth. How far from earth is it at one minute from time $t = 0$?

9. Solve $\frac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. It is OK to leave your answer as a definite integral.

10. Solve $\frac{dy}{dx} = e^x + x$ and $y(0) = 10$.

11. Solve $x' = \frac{1}{x^2}$, $x(1) = 1$.

12. Solve $x' = \frac{1}{\cos(x)}$, $x(0) = \frac{\pi}{2}$.

13. Sid is in a car travelling at speed $10t + 70$ miles per hour away from Las Vegas, where t is in hours. At $t = 0$, Sid is 10 miles away from Vegas. How far from Vegas is Sid 2 hours later?

14. Solve $y' = y^n$, $y(0) = 1$, where n is a positive integer. Hint: You have to consider different cases.

8.3 Slope fields

Note: you might find the software *DFIELD* and *PPLANE* useful. You can download the programs at http://math.rice.edu/~dfield/dfpp.html. These used to be available as in-browser Java applets, but due to changes in Java security settings, you need to download the programs and run them locally. Both Java and MATLAB versions are available.

Another option is the IODE software which accompanies the lecture notes by Jiří Lebl from which we've borrowed the text for this chapter.

As we said, the general first order equation we are studying looks like

$$y' = f(x, y).$$

In general, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behaviour of the solutions, or if we could find approximate solutions.

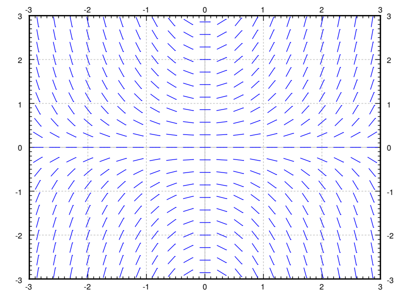

Figure 8.3.1: Slope field for the equation $y' = xy$

Slope fields

Suppose we are able to solve a first order equation of the form $y' = f(x, y)$, obtaining a solution $y = g(x)$. Differential calculus tells us that $y' = g'(x)$ gives us the slope of the tangent line to the curve $y = g(x)$ at the point $(x, g(x))$. Thus, the equation $y' = f(x, y)$ gives you a slope at each point in the (x, y)-plane. We can plot the slope at lots of points as a short line through the point (x, y) with the slope $f(x, y)$. See Figure 8.3.1.

We call this picture the **slope field** of the equation. If we are given a specific initial condition $y(x_0) = y_0$, we can look at the location (x_0, y_0) and follow the slopes. See Figure 8.3.2.

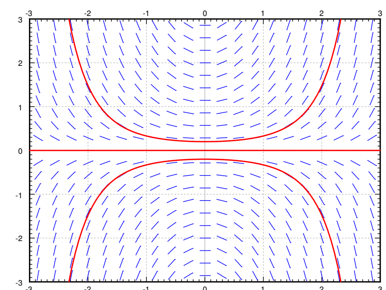

Figure 8.3.2: Slope field of $y' = xy$ with a graph of solutions satisfying $y(0) = 0.2$, $y(0) = 0$, and $y(0) = -0.2$.

By looking at the slope field we can get a lot of information about the behaviour of solutions. For example, in Figure 8.3.2 we can see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. Note that a small change in the initial condition causes quite different behaviour. On the other hand, plotting a few solutions of the equation $y' = -y$, we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. See Figure 8.3.3.

Existence and uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

(i) Does a solution *exist*?

(ii) Is the solution *unique* (if it exists)?

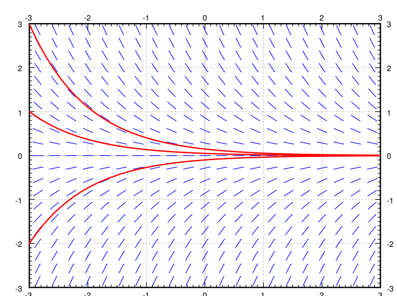

Figure 8.3.3: Slope field of $y' = -y$ with a graph of a few solutions.

What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

Example 8.3.1 An initial value problem with no solution

Attempt to solve:

$$y' = \frac{1}{x}, \qquad y(0) = 0.$$

Solution Integrate to find the general solution $y = \ln|x| + C$. Note that the solution does not exist at $x = 0$. See Figure 8.3.4.

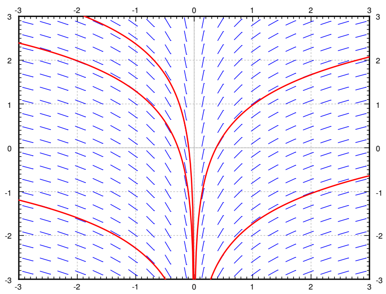

Figure 8.3.4: Slope field of $y' = \dfrac{1}{x}$.

Example 8.3.2 An initial value problem without a unique solution

Solve:

$$y' = 2\sqrt{|y|}, \qquad y(0) = 0.$$

Solution See Figure 8.3.5. Note that $y = 0$ is a solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0, \\ -x^2 & \text{if } x < 0. \end{cases}$$

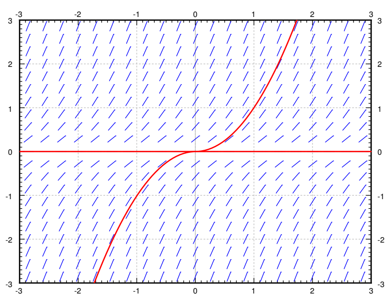

Figure 8.3.5: Slope field of $y' = 2\sqrt{|y|}$ with two solutions satisfying $y(0) = 0$.

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as Picard's theorem

Theorem 8.3.1 Picard's theorem on existence and uniqueness

If $f(x, y)$ is continuous (as a function of two variables) and $\frac{\partial f}{\partial y}$ exists and is continuous near some (x_0, y_0), then a solution to

$$y' = f(x, y), \qquad y(x_0) = y_0,$$

exists (at least for some small interval of x's) and is unique.

Picard's Theorem is named after the French mathematician Charles Émile Picard (1856 – 1941)

Note that the problems $y' = \dfrac{1}{x}$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

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Example 8.3.3 An initial value problem with a "finite time" solution

For some constant A, solve:

$$y' = y^2, \qquad y(0) = A.$$

Solution We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = \dfrac{1}{y^2}$, so $x = \dfrac{-1}{y} + C$, so $y = \frac{1}{C-x}$. If $y(0) = A$, then $C = \dfrac{1}{A}$ so

$$y = \frac{1}{\frac{1}{A} - x}.$$

If $A = 0$, then $y = 0$ is a solution.

For example, when $A = 1$ the solution "blows up" at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds "globally" unlike for the equation $y' = y^2$.

Exercises 8.3

Problems

1. Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?

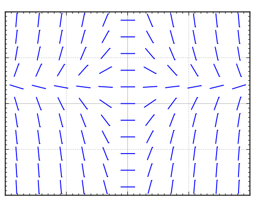
2. Sketch slope field for $y' = x^2$.

3. Sketch slope field for $y' = y^2$.

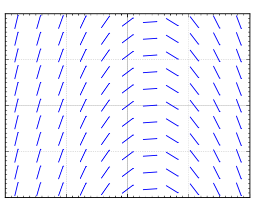
4. Is it possible to solve the equation $y' = \dfrac{xy}{\cos x}$ for $y(0) = 1$? Justify.

5. Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.

6. Match the following equations to their slope fields.
(i) $y' = 1 - x$
(ii) $y' = x - 2y$
(iii) $y' = x(1 - y)$
Justify.



(a)



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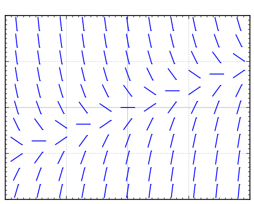


(c)

7. (Challenging) Take $y' = f(x, y)$, $y(0) = 0$, where $f(x, y) > 1$ for all x and y. If the solution exists for all x, can you say what happens to $y(x)$ as x goes to positive infinity? Explain.

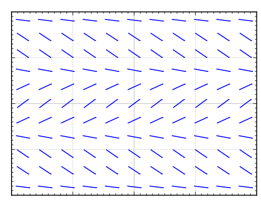
8. (Challenging) Take $(y - x)y' = 0$, $y(0) = 0$. a) Find two distinct solutions. b) Explain why this does not violate Picard's theorem.

9. Sketch the slope field of $y' = y^3$. Can you visually find the solution that satisfies $y(0) = 0$?

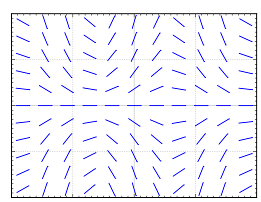
10. Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique?

11. Is it possible to solve $y' = \dfrac{x}{x^2 - 1}$ for $y(1) = 0$?

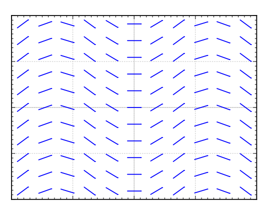
12. Match the following equations to their slope fields:
(i) $y' = \sin x$
(ii) $y' = \cos y$
(iii) $y' = y\cos(x)$
Justify.



(a)



(b)



(c)

8.4 Separable equations

When a differential equation is of the form $y' = f(x)$, we can just integrate: $y = \int f(x)\, dx + C$. Unfortunately this method no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields

$$y = \int f(x, y)\, dx + C.$$

Notice the dependence on y in the integral.

Separable equations

Let us suppose that the equation is **separable**. That is, let us consider

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the **Leibniz notation**

$$\frac{dy}{dx} = f(x)g(y).$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)\, dx.$$

Now both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)\, dx + C.$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y.

Example 8.4.1 A separable ODE

Solve the equation

$$y' = xy.$$

SOLUTION First note that $y = 0$ is a solution, so assume $y \neq 0$ from now on. Write the equation as $\frac{dy}{dx} = xy$, then

$$\int \frac{dy}{y} = \int x\, dx + C.$$

We compute the antiderivatives to get

$$\ln |y| = \frac{x^2}{2} + C.$$

Or

$$|y| = e^{\frac{x^2}{2} + C} = e^{\frac{x^2}{2}} e^C = De^{\frac{x^2}{2}},$$

where $D > 0$ is some constant. Because $y = 0$ is a solution and because of the absolute value we actually can write:

$$y = De^{\frac{x^2}{2}},$$

for any number D (including zero or negative).

We check:
$$y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right) = xy.$$
It works!

We should be a little bit more careful with this method. You may be worried that we were integrating in two different variables. We seemed to be doing a different operation to each side. Let us work this method out more rigorously. Take
$$\frac{dy}{dx} = f(x)g(y).$$
We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$!
$$\frac{1}{g(y)}\,\frac{dy}{dx} = f(x).$$
We integrate both sides with respect to x.
$$\int \frac{1}{g(y)}\,\frac{dy}{dx}\,dx = \int f(x)\,dx + C.$$
We can use the change of variables formula.
$$\int \frac{1}{g(y)}\,dy = \int f(x)\,dx + C.$$
And we are done.

Implicit solutions

It is clear that we might sometimes get stuck even if we can do the integration. For example, take the separable equation
$$y' = \frac{xy}{y^2+1}.$$
We separate variables,
$$\frac{y^2+1}{y}\,dy = \left(y + \frac{1}{y}\right)\,dy = x\,dx.$$
We integrate to get
$$\frac{y^2}{2} + \ln|y| = \frac{x^2}{2} + C,$$
or perhaps the easier looking expression (where $D = 2C$)
$$y^2 + 2\ln|y| = x^2 + D.$$
It is not easy to find the solution explicitly as it is hard to solve for y. We, therefore, leave the solution in this form and call it an **implicit solution**. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x to get
$$y'\left(2y + \frac{2}{y}\right) = 2x.$$
It is simple to see that the differential equation holds. If you want to compute values for y, you might have to be tricky. For example, you can graph x as a

function of y, and then flip your paper. Computers are also good at some of these tricks.

We note that the above equation also has the solution $y = 0$. The general solution is $y^2 + 2\ln|y| = x^2 + C$ together with $y = 0$. These outlying solutions such as $y = 0$ are sometimes called *singular solutions*.

Example 8.4.2 An example with initial conditions

Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2$, $y(1) = 0$.

SOLUTION First factor the right hand side to obtain

$$x^2y' = (1 - x^2)(1 + y^2).$$

Separate variables, integrate, and solve for y.

$$\begin{aligned} \frac{y'}{1+y^2} &= \frac{1-x^2}{x^2}, \\ \frac{y'}{1+y^2} &= \frac{1}{x^2} - 1, \\ \arctan(y) &= \frac{-1}{x} - x + C, \\ y &= \tan\left(\frac{-1}{x} - x + C\right). \end{aligned}$$

Now solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $2 + \pi$, etc...). The solution we are seeking is, therefore,

$$y = \tan\left(\frac{-1}{x} - x + 2\right).$$

Example 8.4.3 Cooling a cup of coffee

Bob made a cup of coffee, and Bob likes to drink coffee only once it will not burn him at 60 degrees. Initially at time $t = 0$ minutes, Bob measured the temperature and the coffee was 89 degrees Celsius. One minute later, Bob measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Bob start drinking?

SOLUTION Let T be the temperature of the coffee, and let A be the ambient (room) temperature. **Newton's law of cooling** states that the rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

for some constant k. For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants)

$$\begin{aligned} \frac{1}{T-A}\frac{dT}{dt} &= -k, \\ \ln(T - A) &= -kt + C, \qquad \text{(note that } T - A > 0\text{)} \\ T - A &= D\,e^{-kt}, \\ T &= A + D\,e^{-kt}. \end{aligned}$$

That is, $T = 22 + D\,e^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67\,e^{-kt}$. The second condition says $85 = T(1) = 22 + 67\,e^{-k}$. Solving for k we get $k = -\ln \frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. That is, we solve $60 = 22 + 67e^{-0.0616t}$ to get $t = -\frac{\ln \frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Bob can begin to drink the coffee at just over 9 minutes from the time Bob made it. That is probably about the amount of time it took us to calculate how long it would take.

Example 8.4.4 Finding all possible solutions

Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

Solution First note that $y = 0$ is a solution (a singular solution). So assume that $y \neq 0$ and write

$$\begin{aligned}\frac{-3}{y^2}y' &= x,\\ \frac{3}{y} &= \frac{x^2}{2} + C,\\ y &= \frac{3}{x^2/2 + C} = \frac{6}{x^2 + 2C}.\end{aligned}$$

Exercises 8.4

Problems

1. Solve $y' = \dfrac{x}{y}$.

2. Solve $y' = x^2 y$.

3. Solve $\dfrac{dx}{dt} = (x^2 - 1)\,t$, for $x(0) = 0$.

4. Solve $\dfrac{dx}{dt} = x\ \sin(t)$, for $x(0) = 1$.

5. Solve $\dfrac{dy}{dx} = xy + x + y + 1$. Hint: Factor the right hand side.

6. Solve $xy' = y + 2x^2 y$, where $y(1) = 1$.

7. Solve $\dfrac{dy}{dx} = \dfrac{y^2 + 1}{x^2 + 1}$, for $y(0) = 1$.

8. Find an implicit solution for $\dfrac{dy}{dx} = \dfrac{x^2 + 1}{y^2 + 1}$, for $y(0) = 1$.

9. Find an explicit solution for $y' = xe^{-y}$, $y(0) = 1$.

10. Find an explicit solution for $xy' = e^{-y}$, for $y(1) = 1$.

11. Find an explicit solution for $y' = ye^{-x^2}$, $y(0) = 1$. It is all right to leave a definite integral in your answer.

12. Suppose a cup of coffee is at 100 degrees Celsius at time $t = 0$, it is at 70 degrees at $t = 10$ minutes, and it is at 50 degrees at $t = 20$ minutes. Compute the ambient temperature.

13. Solve $y' = 2xy$.

14. Solve $x' = 3xt^2 - 3t^2$, $x(0) = 2$.

15. Find an implicit solution for $x' = \frac{1}{3x^2+1}$, $x(0) = 1$.

16. Find an explicit solution to $xy' = y^2$, $y(1) = 1$.

17. Find an implicit solution to $y' = \frac{\sin(x)}{\cos(y)}$.

18. Take Example 8.4.3 with the same numbers: 89 degrees at $t = 0$, 85 degrees at $t = 1$, and ambient temperature of 22 degrees. Suppose these temperatures were measured with precision of ± 0.5 degrees. Given this imprecision, the time it takes the coffee to cool to (exactly) 60 degrees is also only known in a certain range. Find this range. Hint: Think about what kind of error makes the cooling time longer and what shorter.

8.5 Linear equations and the integrating factor

One of the most important types of equations we will learn how to solve are the so-called **linear equations** In this lecture we focus on the **first order linear equation**. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \tag{8.4}$$

Here the word "linear" means linear in y and y'; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations.

The trick is to rewrite the left hand side of (8.4) as a derivative of a product of y with another function. To this end we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}\Big[r(x)y\Big].$$

This is the left hand side of (8.4) multiplied by $r(x)$. So if we multiply (8.4) by $r(x)$, we obtain

$$\frac{d}{dx}\Big[r(x)y\Big] = r(x)f(x).$$

Now we integrate both sides. The right hand side does not depend on y and the left hand side is written as a derivative of a function. Afterwards, we solve for y. The function $r(x)$ is called the **integrating factor** and the method is called the **integrating factor method**.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)\,dx}.$$

We compute:

$$\begin{aligned}
y' + p(x)y &= f(x), \\
e^{\int p(x)\,dx}y' + e^{\int p(x)\,dx}p(x)y &= e^{\int p(x)\,dx}f(x), \\
\frac{d}{dx}\left[e^{\int p(x)\,dx}y\right] &= e^{\int p(x)\,dx}f(x), \\
e^{\int p(x)\,dx}y &= \int e^{\int p(x)\,dx}f(x)\,dx + C, \\
y &= e^{-\int p(x)\,dx}\left(\int e^{\int p(x)\,dx}f(x)\,dx + C\right).
\end{aligned}$$

Of course, to get a closed form formula for y, we need to be able to find a closed form formula for the integrals appearing above.

Example 8.5.1 A linear equation with a closed form solution

Solve

$$y' + 2xy = e^{x-x^2}, \qquad y(0) = -1.$$

Solution First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)\,dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$

to get

$$e^{x^2}y' + 2xe^{x^2}y = e^{x-x^2}e^{x^2},$$
$$\frac{d}{dx}\left[e^{x^2}y\right] = e^x.$$

We integrate

$$e^{x^2}y = e^x + C,$$
$$y = e^{x-x^2} + Ce^{-x^2}.$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

Exercise: Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above.

A piece of advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \qquad y(x_0) = y_0.$$

Look at the solution and write the integrals as definite integrals.

$$\boxed{y(x) = e^{-\int_{x_0}^{x} p(s)\,ds}\left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)\,ds} f(t)\,dt + y_0\right).} \tag{8.5}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

Exercise: Check that $y(x_0) = y_0$ in formula (8.5).

Exercise: Write the solution of the following problem as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.

$$y' + y = e^{x^2-x}, \qquad y(0) = 10.$$

Remark: Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is always an explicit formula (8.5) for the solution. Second, it follows from the formula (8.5) that if $p(x)$ and $f(x)$ are continuous on some interval (a, b), then the solution $y(x)$ exists and is differentiable on (a, b). Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to Theorem 8.3.1.

Let us discuss a common simple application of linear equations. This type of problem is used often in real life. For example, linear equations are used in figuring out the concentration of chemicals in bodies of water (rivers and lakes).

Example 8.5.2 An application of linear ODEs

A 100 litre tank contains 10 kilograms of salt dissolved in 60 litres of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per litre is flowing in at the rate of 5 litres a minute. The solution in the tank is well stirred and flows out at a rate of 3 litres a minute. How much salt is in the tank when the tank is full?

Solution Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in} \times \text{concentration in})\Delta t - (\text{rate out} \times \text{concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \to 0$ we see that

$$\frac{dx}{dt} = (\text{rate in} \times \text{concentration in}) - (\text{rate out} \times \text{concentration out}).$$

In our example, we have

$$\begin{aligned}
\text{rate in} &= 5,\\
\text{concentration in} &= 0.1,\\
\text{rate out} &= 3,\\
\text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60 + (5-3)t}.
\end{aligned}$$

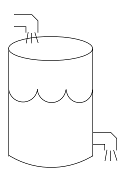


Figure 8.5.1: The tank in Example 8.5.2

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \times 0.1) - \left(3\frac{x}{60+2t}\right).$$

Or in the form (8.4)

$$\frac{dx}{dt} + \frac{3}{60+2t}x = 0.5.$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t}dt\right) = \exp\left(\frac{3}{2}\ln(60+2t)\right) = (60+2t)^{3/2}.$$

We multiply both sides of the equation to get

$$(60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x = 0.5(60+2t)^{3/2},$$

and reversing the product rule gives us

$$\frac{d}{dt}\left[(60+2t)^{3/2}x\right] = 0.5(60+2t)^{3/2},$$

so

$$(60+2t)^{3/2}x = \int 0.5(60+2t)^{3/2}dt + C.$$

Thus,

$$\begin{aligned}
x &= (60+2t)^{-3/2}\int \frac{(60+2t)^{3/2}}{2}dt + C(60+2t)^{-3/2},\\
&= (60+2t)^{-3/2}\frac{1}{10}(60+2t)^{5/2} + C(60+2t)^{-3/2},\\
&= \frac{60+2t}{10} + C(60+2t)^{-3/2}.
\end{aligned}$$

We need to find C. We know that at $t = 0$, $x = 10$. So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2},$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We are interested in x when the tank is full. So we note that the tank is full when $60 + 2t = 100$, or when $t = 20$. So

$$x(20) = \frac{60+40}{10} + C(60+40)^{-3/2} \approx 10 + 1859.03(100)^{-3/2} \approx 11.86.$$

The concentration at the end is approximately 0.1186 $^{\text{kg}}/_{\text{litre}}$ and we started with $\frac{1}{6}$ or 0.167 $^{\text{kg}}/_{\text{litre}}$.

Exercises 8.5

Problems

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

1. Solve $y' + xy = x$.

2. Solve $y' + 6y = e^x$.

3. Solve $y' + 3x^2 y = \sin(x)\, e^{-x^3}$, with $y(0) = 1$.

4. Solve $y' + \cos(x)y = \cos(x)$.

5. Solve $\frac{1}{x^2+1}\, y' + xy = 3$, with $y(0) = 0$.

6. Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 litres per hour. The first lake contains 100 thousand litres of water and the second lake contains 200 thousand litres of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream. a) Find the concentration of toxic substance as a function of time in both lakes. b) When will the concentration in the first lake be below 0.001 kg per litre? c) When will the concentration in the second lake be maximal?

7. **Newton's law of cooling** states that $\frac{dx}{dt} = -k(x - A)$ where x is the temperature, t is time, A is the ambient temperature, and $k > 0$ is a constant. Suppose that $A = A_0 \cos(\omega t)$ for some constants A_0 and ω. That is, the ambient temperature oscillates (for example night and day temperatures). a) Find the general solution. b) In the long term, will the initial conditions make much of a difference? Why or why not?

8. Initially 5 grams of salt are dissolved in 20 litres of water. Brine with concentration of salt 2 grams of salt per litre is added at a rate of 3 litres a minute. The tank is mixed well and is drained at 3 litres a minute. How long does the process have to continue until there are 20 grams of salt in the tank?

9. Initially a tank contains 10 litres of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 litre per minute. The water is mixed well and drained at 1 litre per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?

10. Solve $y' + 3x^2 y = x^2$.

11. Solve $y' + 2\sin(2x)y = 2\sin(2x)$, $y(\pi/2) = 3$.

12. Suppose a water tank is being pumped out at 3 L/min. The water tank starts at 10 L of clean water. Water with toxic substance is flowing into the tank at 2 L/min, with concentration $20t$ g/L at time t. When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?

13. Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is, suppose that $\frac{dP}{dt} = (2 - 0.1t)P$. If $P(0) = 1000$, find the population at $t = 5$.

8.6 Numerical methods: Euler's method

At this point it may be good to first try the Lab II and/or Project II from the IODE website: http://www.math.uiuc.edu/iode/materials.html. (This is completely optional, and you're free to look for your own software solutions online, or try using Maple or similar software. But it is generally a good idea to have the computer's help when exploring Euler's method.)

As we said before, unless $f(x, y)$ is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

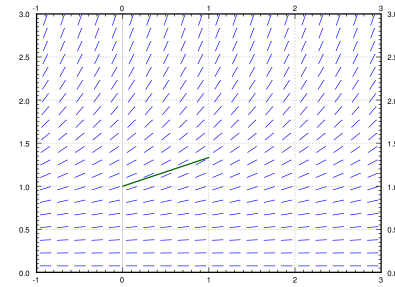
$$y' = f(x, y), \qquad y(x_0) = y_0.$$

What if we want to find the value of the solution at some particular x? Or perhaps we want to produce a graph of the solution to inspect the behaviour. In this section we will learn about the basics of numerical approximation of solutions.

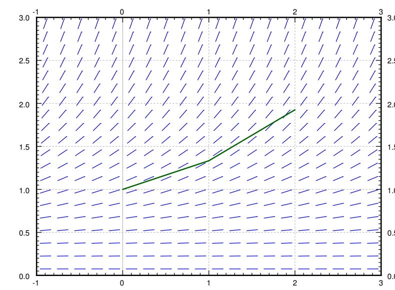
The simplest method for approximating a solution is **Euler's method**

.

It works as follows: We take x_0 and compute the slope $k = f(x_0, y_0)$. The slope is the change in y per unit change in x. We follow the line for an interval of length h on the x axis. Hence if $y = y_0$ at x_0, then we will say that y_1 (the approximate value of y at $x_1 = x_0 + h$) will be $y_1 = y_0 + hk$. Rinse, repeat! That is, compute x_2 and y_2 using x_1 and y_1. For an example of the first two steps of the method see Figure 8.6.1.



Step 1



Step 2

Figure 8.6.1: First two steps of Euler's method with $h = 1$ for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

More abstractly, for any $i = 1, 2, 3, \ldots$, we compute

$$x_{i+1} = x_i + h, \qquad y_{i+1} = y_i + h\,f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure 8.6.2 for the plot of the real solution and the approximation.

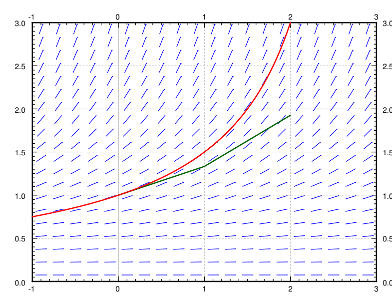


Figure 8.6.2: Two steps of Euler's method (step size 1) and the exact solution for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$.

Let us see what happens with the equation $y' = \dfrac{y^2}{3}$, $y(0) = 1$. Let us try to approximate $y(2)$ using Euler's method. In Figures 8.6.1 and 8.6.2 we have graphically approximated $y(2)$ with step size 1. With step size 1 we have $y(2) \approx 1.926$. The real answer is 3. So we are approximately 1.074 off. Let us halve the step size. Computing y_4 with $h = 0.5$, we find that $y(2) \approx 2.209$, so an error of about 0.791. Table 8.1 gives the values computed for various parameters.

Exercise: Solve this equation exactly and show that $y(2) = 3$.

The difference between the actual solution and the approximate solution we will call the error. We will usually talk about just the size of the error and we do not care much about its sign. The main point is, that we usually do not know the real solution, so we only have a vague understanding of the error. If we knew the error exactly …what is the point of doing the approximation?

We notice that except for the first few times, every time we halved the interval the error approximately halved. This halving of the error is a general feature of Euler's method as it is a **first order method**. In the IODE Project II you are asked to implement a **second order method**. A second order method reduces the error to approximately one quarter every time we halve the interval (second order as $\frac{1}{4} = \frac{1}{2} \times \frac{1}{2}$).

Euler's Method is named after the Swiss mathematician Leonhard Paul Euler (1707 – 1783). Do note the correct pronunciation of the name sounds more like "oiler."

To get the error to be within 0.1 of the answer we had to already do 64 steps. To get it to within 0.01 we would have to halve another three or four times,

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

Table 8.1: Euler's method approximation of $y(2)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

h	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004
0.015625	50.40121
0.0078125	87.75769

Figure 8.6.3: Attempts to use Euler's to approximate $y(3)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

meaning doing 512 to 1024 steps. That is quite a bit to do by hand. The improved Euler method from IODE Project II should quarter the error every time we halve the interval, so we would have to approximately do half as many "halvings" to get the same error. This reduction can be a big deal. With 10 halvings (starting at $h = 1$) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more difficult to compute than $\frac{y^2}{3}$). Then the difference is 32 seconds versus about 17 minutes. Note: We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

Note that in practice we do not know how large the error is! How do we know what is the right step size? Well, essentially we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

Exercise: In the table above, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?

Let us talk a little bit more about the example $y' = \frac{y^2}{3}$, $y(0) = 1$. Suppose that instead of the value $y(2)$ we wish to find $y(3)$. The results of this effort are listed in Table 8.6.3 for successive halvings of h. What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at $x = 3$. In fact, the solution goes to infinity when you approach $x = 3$.

Another case where things go bad is if the solution oscillates wildly near some point. Such an example is given in IODE Project II. The solution may exist at all points, but even a much better numerical method than Euler would need an insanely small step size to approximate the solution with reasonable precision. And computers might not be able to easily handle such a small step size.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge-Kutta method. That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16 (it is fourth order as $\frac{1}{16} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$).

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

- Computational time: Each step takes computer time. Even if the function f is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.

- Roundoff errors: Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse. There is a certain optimum step size such that the precision increases as we approach it, but then starts getting worse as we make our step size smaller still. Trouble is: this optimum may be hard to find.

- Stability: Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called *stiff*. In the worst case, the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers seem to have stabilized after successive halving, does not mean that we must have the right answer.

We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area for engineers and mathematicians. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

Exercises 8.6

Problems

1. Consider $\frac{dx}{dt} = (2t - x)^2$, $x(0) = 2$. Use Euler's method with step size $h = 0.5$ to approximate $x(1)$.

2. Consider $\frac{dx}{dt} = t - x$, $x(0) = 1$. a) Use Euler's method with step sizes $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ to approximate $x(1)$. b) Solve the equation exactly. c) Describe what happens to the errors for each h you used. That is, find the factor by which the error changed each time you halved the interval.

3. Approximate the value of e by looking at the initial value problem $y' = y$ with $y(0) = 1$ and approximating $y(1)$ using Euler's method with a step size of 0.2.

4. Example of numerical instability: Take $y' = -5y$, $y(0) = 1$. We know that the solution should decay to zero as x grows. Using Euler's method, start with $h = 1$ and compute y_1, y_2, y_3, y_4 to try to approximate $y(4)$. What happened? Now halve the interval. Keep halving the interval and approximating $y(4)$ until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.

5. Let $x' = \sin(xt)$, and $x(0) = 1$. Approximate $x(1)$ using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.

6. Let $x' = 2t$, and $x(0) = 0$. a) Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1. b) Solve exactly, and compute the errors. c) Compute the factor by which the errors changed.

7. Let $x' = xe^{xt+1}$, and $x(0) = 0$. (a) Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1. (b) Guess an exact solution based on part (a) and compute the errors.

9: Curves in the Plane

We have explored functions of the form $y = f(x)$ closely throughout this text. We have explored their limits, their derivatives and their antiderivatives; we have learned to identify key features of their graphs, such as relative maxima and minima, inflection points and asymptotes; we have found equations of their tangent lines, the areas between portions of their graphs and the x-axis, and the volumes of solids generated by revolving portions of their graphs about a horizontal or vertical axis.

Despite all this, the graphs created by functions of the form $y = f(x)$ are limited. Since each x-value can correspond to only 1 y-value, common shapes like circles cannot be fully described by a function in this form. Fittingly, the "vertical line test" excludes vertical lines from being functions of x, even though these lines are important in mathematics.

In this chapter we'll explore new ways of drawing curves in the plane. We'll still work within the framework of functions, as an input will still only correspond to one output. However, our new techniques of drawing curves will render the vertical line test pointless, and allow us to create important – and beautiful – new curves. Once these curves are defined, we'll apply the concepts of calculus to them, continuing to find equations of tangent lines and the areas of enclosed regions.

9.1 Conic Sections

The ancient Greeks recognized that interesting shapes can be formed by intersecting a plane with a *double napped* cone (i.e., two identical cones placed tip–to–tip as shown in the following figures). As these shapes are formed as sections of conics, they have earned the official name "conic sections."

The three "most interesting" conic sections are given in the top row of Figure 9.1.1. They are the parabola, the ellipse (which includes circles) and the hyperbola. In each of these cases, the plane does not intersect the tips of the cones (usually taken to be the origin).

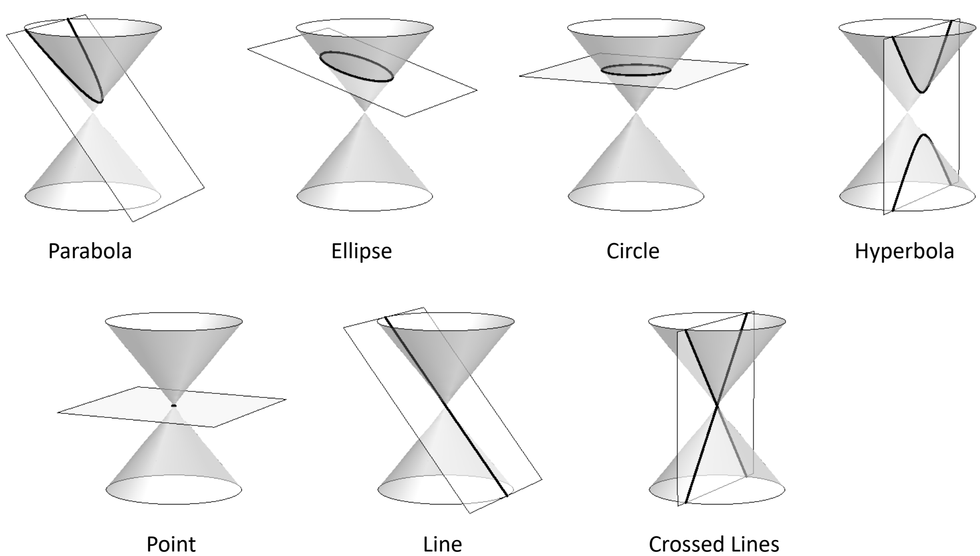


Figure 9.1.1: Conic Sections

When the plane does contain the origin, three **degenerate** cones can be formed as shown the bottom row of Figure 9.1.1: a point, a line, and crossed lines. We focus here on the nondegenerate cases.

While the above geometric constructs define the conics in an intuitive, visual way, these constructs are not very helpful when trying to analyze the shapes algebraically or consider them as the graph of a function. It can be shown that all conics can be defined by the general second–degree equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0.$$

While this algebraic definition has its uses, most find another geometric perspective of the conics more beneficial.

Each nondegenerate conic can be defined as the **locus**, or set, of points that satisfy a certain distance property. These distance properties can be used to generate an algebraic formula, allowing us to study each conic as the graph of a function.

Parabolas

Definition 9.1.1 Parabola

A **parabola** is the locus of all points equidistant from a point (called a **focus**) and a line (called the **directrix**) that does not contain the focus.

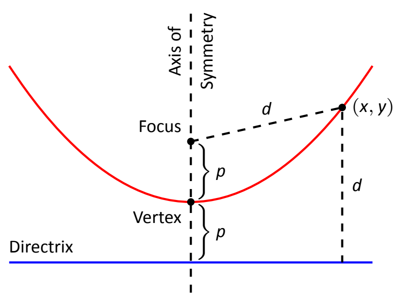


Figure 9.1.2: Illustrating the definition of the parabola and establishing an algebraic formula.

Figure 9.1.2 illustrates this definition. The point halfway between the focus and the directrix is the **vertex**. The line through the focus, perpendicular to the directrix, is the **axis of symmetry**, as the portion of the parabola on one side of this line is the mirror–image of the portion on the opposite side.

The definition leads us to an algebraic formula for the parabola. Let $P = (x, y)$ be a point on a parabola whose focus is at $F = (0, p)$ and whose directrix is at $y = -p$. (We'll assume for now that the focus lies on the y-axis; by placing the focus p units above the x-axis and the directrix p units below this axis, the vertex will be at $(0, 0)$.)

We use the Distance Formula to find the distance d_1 between F and P:

$$d_1 = \sqrt{(x-0)^2 + (y-p)^2}.$$

The distance d_2 from P to the directrix is more straightforward:

$$d_2 = y - (-p) = y + p.$$

These two distances are equal. Setting $d_1 = d_2$, we can solve for y in terms of x:

$$\begin{aligned} d_1 &= d_2 \\ \sqrt{x^2 + (y-p)^2} &= y + p \end{aligned}$$

Now square both sides.

$$\begin{aligned} x^2 + (y-p)^2 &= (y+p)^2 \\ x^2 + y^2 - 2yp + p^2 &= y^2 + 2yp + p^2 \\ x^2 &= 4yp \\ y &= \frac{1}{4p}x^2. \end{aligned}$$

The geometric definition of the parabola has led us to the familiar quadratic function whose graph is a parabola with vertex at the origin. When we allow the vertex to not be at $(0,0)$, we get the following standard form of the parabola.

Key Idea 9.1.1 General Equation of a Parabola

1. **Vertical Axis of Symmetry:** The equation of the parabola with vertex at (h,k) and directrix $y=k-p$ in standard form is

$$y=\frac{1}{4p}(x-h)^2+k.$$

The focus is at $(h,k+p)$.

2. **Horizontal Axis of Symmetry:** The equation of the parabola with vertex at (h,k) and directrix $x=h-p$ in standard form is

$$x=\frac{1}{4p}(y-k)^2+h.$$

The focus is at $(h+p,k)$.

Note: p is not necessarily a positive number.

Example 9.1.1 Finding the equation of a parabola

Give the equation of the parabola with focus at $(1,2)$ and directrix at $y=3$.

Solution The vertex is located halfway between the focus and directrix, so $(h,k)=(1,2.5)$. This gives $p=-0.5$. Using Key Idea 9.1.1 we have the equation of the parabola as

$$y=\frac{1}{4(-0.5)}(x-1)^2+2.5=-\frac{1}{2}(x-1)^2+2.5.$$

The parabola is sketched in Figure 9.1.3.

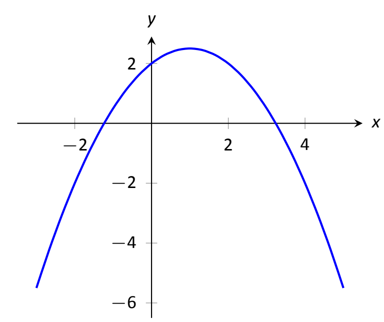


Figure 9.1.3: The parabola described in Example 9.1.1.

Example 9.1.2 Finding the focus and directrix of a parabola

Find the focus and directrix of the parabola $x=\frac{1}{8}y^2-y+1$. The point $(7,12)$ lies on the graph of this parabola; verify that it is equidistant from the focus and directrix.

Solution We need to put the equation of the parabola in its general form. This requires us to complete the square:

$$\begin{aligned} x&=\frac{1}{8}y^2-y+1\\ &=\frac{1}{8}\left(y^2-8y+8\right)\\ &=\frac{1}{8}\left(y^2-8y+16-16+8\right)\\ &=\frac{1}{8}\left((y-4)^2-8\right)\\ &=\frac{1}{8}(y-4)^2-1. \end{aligned}$$

Hence the vertex is located at $(-1,4)$. We have $\frac{1}{8}=\frac{1}{4p}$, so $p=2$. We conclude that the focus is located at $(1,4)$ and the directrix is $x=-3$. The parabola is

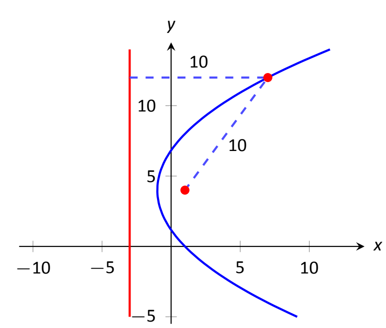


Figure 9.1.4: The parabola described in Example 9.1.2. The distances from a point on the parabola to the focus and directrix is given.

graphed in Figure 9.1.4, along with its focus and directrix.

The point $(7, 12)$ lies on the graph and is $7 - (-3) = 10$ units from the directrix. The distance from $(7, 12)$ to the focus is:

$$\sqrt{(7-1)^2 + (12-4)^2} = \sqrt{100} = 10.$$

Indeed, the point on the parabola is equidistant from the focus and directrix.

Reflective Property

One of the fascinating things about the nondegenerate conic sections is their reflective properties. Parabolas have the following reflective property:

> Any ray emanating from the focus that intersects the parabola reflects off along a line perpendicular to the directrix.

This is illustrated in Figure 9.1.5. The following theorem states this more rigorously.

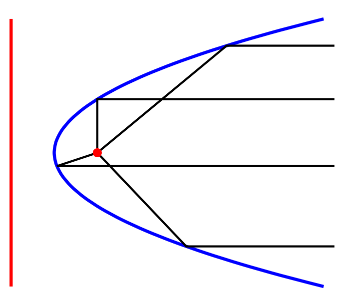


Figure 9.1.5: Illustrating the parabola's reflective property.

Theorem 9.1.1 Reflective Property of the Parabola

Let P be a point on a parabola. The tangent line to the parabola at P makes equal angles with the following two lines:

1. The line containing P and the focus F, and
2. The line perpendicular to the directrix through P.

Because of this reflective property, paraboloids (the 3D analogue of parabolas) make for useful flashlight reflectors as the light from the bulb, ideally located at the focus, is reflected along parallel rays. Satellite dishes also have paraboloid shapes. Signals coming from satellites effectively approach the dish along parallel rays. The dish then *focuses* these rays at the focus, where the sensor is located.

Ellipses

Definition 9.1.2 Ellipse

An **ellipse** is the locus of all points whose sum of distances from two fixed points, each a **focus** of the ellipse, is constant.

An easy way to visualize this construction of an ellipse is to pin both ends of a string to a board. The pins become the foci. Holding a pencil tight against the string places the pencil on the ellipse; the sum of distances from the pencil to the pins is constant: the length of the string. See Figure 9.1.6.

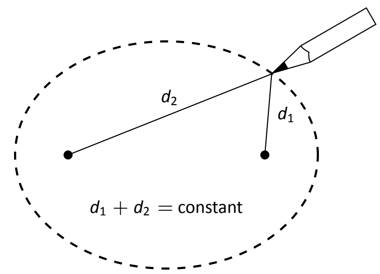


Figure 9.1.6: Illustrating the construction of an ellipse with pins, pencil and string.

We can again find an algebraic equation for an ellipse using this geometric definition. Let the foci be located along the x-axis, c units from the origin. Let these foci be labelled as $F_1 = (-c, 0)$ and $F_2 = (c, 0)$. Let $P = (x, y)$ be a point on the ellipse. The sum of distances from F_1 to P (d_1) and from F_2 to P (d_2) is a constant d. That is, $d_1 + d_2 = d$. Using the Distance Formula, we have

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = d.$$

Using a fair amount of algebra can produce the following equation of an ellipse (note that the equation is an implicitly defined function; it has to be, as an ellipse fails the Vertical Line Test):

$$\frac{x^2}{\left(\frac{d}{2}\right)^2} + \frac{y^2}{\left(\frac{d}{2}\right)^2 - c^2} = 1.$$

This is not particularly illuminating, but by making the substitution $a = d/2$ and $b = \sqrt{a^2 - c^2}$, we can rewrite the above equation as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

This choice of a and b is not without reason; as shown in Figure 9.1.7, the values of a and b have geometric meaning in the graph of the ellipse.

In general, the two foci of an ellipse lie on the **major axis** of the ellipse, and the midpoint of the segment joining the two foci is the **center**. The major axis intersects the ellipse at two points, each of which is a **vertex**. The line segment through the center and perpendicular to the major axis is the **minor axis**. The "constant sum of distances" that defines the ellipse is the length of the major axis, i.e., $2a$.

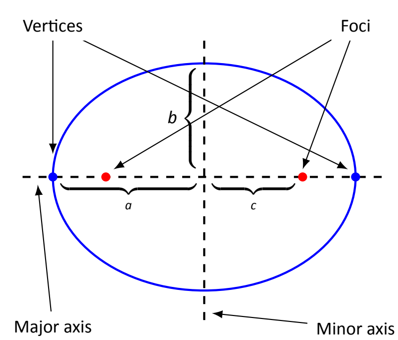


Figure 9.1.7: Labelling the significant features of an ellipse.

Allowing for the shifting of the ellipse gives the following standard equations.

Key Idea 9.1.2 Standard Equation of the Ellipse

The equation of an ellipse centered at (h, k) with major axis of length $2a$ and minor axis of length $2b$ in standard form is:

1. **Horizontal major axis:** $\dfrac{(x-h)^2}{a^2} + \dfrac{(y-k)^2}{b^2} = 1.$

2. **Vertical major axis:** $\dfrac{(x-h)^2}{b^2} + \dfrac{(y-k)^2}{a^2} = 1.$

The foci lie along the major axis, c units from the center, where $c^2 = a^2 - b^2$.

Example 9.1.3 Finding the equation of an ellipse
Find the general equation of the ellipse graphed in Figure 9.1.8.

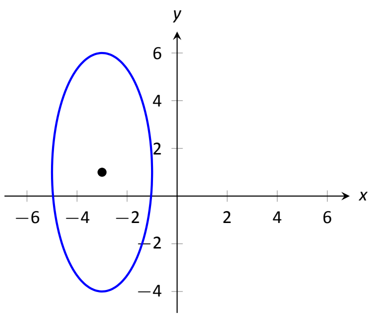


Figure 9.1.8: The ellipse used in Example 9.1.3.

Solution The center is located at $(-3, 1)$. The distance from the center to a vertex is 5 units, hence $a = 5$. The minor axis seems to have length 4, so $b = 2$. Thus the equation of the ellipse is

$$\frac{(x+3)^2}{4} + \frac{(y-1)^2}{25} = 1.$$

Example 9.1.4 Graphing an ellipse
Graph the ellipse defined by $4x^2 + 9y^2 - 8x - 36y = -4$.

Solution It is simple to graph an ellipse once it is in standard form. In order to put the given equation in standard form, we must complete the square with both the x and y terms. We first rewrite the equation by regrouping:

$$4x^2 + 9y^2 - 8x - 36y = -4 \quad \Rightarrow \quad (4x^2 - 8x) + (9y^2 - 36y) = -4.$$

Now we complete the squares.

$$\begin{aligned}(4x^2 - 8x) + (9y^2 - 36y) &= -4\\ 4(x^2 - 2x) + 9(y^2 - 4y) &= -4\\ 4(x^2 - 2x + 1 - 1) + 9(y^2 - 4y + 4 - 4) &= -4\\ 4\big((x-1)^2 - 1\big) + 9\big((y-2)^2 - 4\big) &= -4\\ 4(x-1)^2 - 4 + 9(y-2)^2 - 36 &= -4\\ 4(x-1)^2 + 9(y-2)^2 &= 36\\ \frac{(x-1)^2}{9} + \frac{(y-2)^2}{4} &= 1.\end{aligned}$$

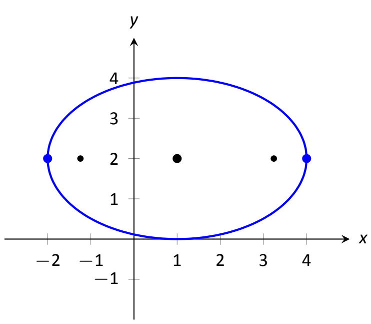


Figure 9.1.9: Graphing the ellipse in Example 9.1.4.

We see the center of the ellipse is at $(1, 2)$. We have $a = 3$ and $b = 2$; the major axis is horizontal, so the vertices are located at $(-2, 2)$ and $(4, 2)$. We find $c = \sqrt{9-4} = \sqrt{5} \approx 2.24$. The foci are located along the major axis, approximately 2.24 units from the center, at $(1 \pm 2.24, 2)$. This is all graphed in Figure 9.1.9 .

Eccentricity

When $a = b$, we have a circle. The general equation becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} = 1 \quad \Rightarrow (x-h)^2 + (y-k)^2 = a^2,$$

the familiar equation of the circle centred at (h, k) with radius a. Since $a = b$, $c = \sqrt{a^2 - b^2} = 0$. The circle has "two" foci, but they lie on the same point, the center of the circle.

Consider Figure 9.1.10, where several ellipses are graphed with $a = 1$. In (a), we have $c = 0$ and the ellipse is a circle. As c grows, the resulting ellipses look less and less circular. A measure of this "noncircularness" is *eccentricity*.

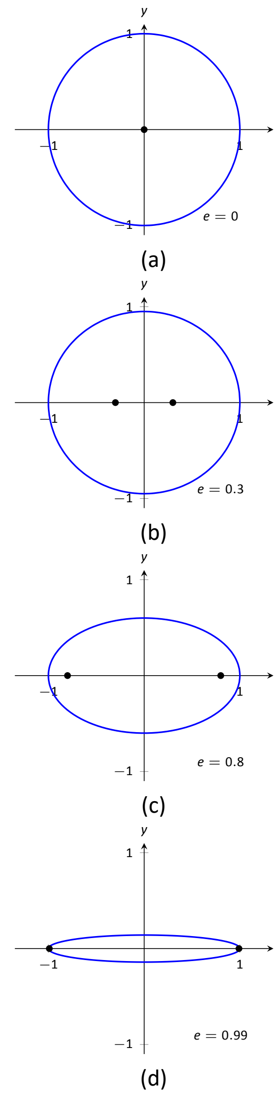


Figure 9.1.10: Understanding the eccentricity of an ellipse.

Definition 9.1.3 Eccentricity of an Ellipse

The eccentricity e of an ellipse is $e = \dfrac{c}{a}$.

The eccentricity of a circle is 0; that is, a circle has no "noncircularness." As c approaches a, e approaches 1, giving rise to a very noncircular ellipse, as seen in Figure 9.1.10 (d).

It was long assumed that planets had circular orbits. This is known to be incorrect; the orbits are elliptical. Earth has an eccentricity of 0.0167 – it has a nearly circular orbit. Mercury's orbit is the most eccentric, with $e = 0.2056$. (Pluto's eccentricity is greater, at $e = 0.248$, the greatest of all the currently known dwarf planets.) The planet with the most circular orbit is Venus, with $e = 0.0068$. The Earth's moon has an eccentricity of $e = 0.0549$, also very circular.

Reflective Property

The ellipse also possesses an interesting reflective property. Any ray emanating from one focus of an ellipse reflects off the ellipse along a line through the other focus, as illustrated in Figure 9.1.11. This property is given formally in the following theorem.

Theorem 9.1.2 Reflective Property of an Ellipse

Let P be a point on a ellipse with foci F_1 and F_2. The tangent line to the ellipse at P makes equal angles with the following two lines:

1. The line through F_1 and P, and
2. The line through F_2 and P.

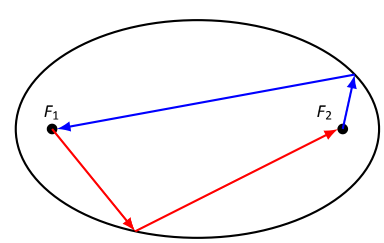


Figure 9.1.11: Illustrating the reflective property of an ellipse.

This reflective property is useful in optics and is the basis of the phenomena experienced in whispering halls.

Hyperbolas

The definition of a hyperbola is very similar to the definition of an ellipse; we essentially just change the word "sum" to "difference."

Definition 9.1.4 Hyperbola

A **hyperbola** is the locus of all points where the absolute value of difference of distances from two fixed points, each a focus of the hyperbola, is constant.

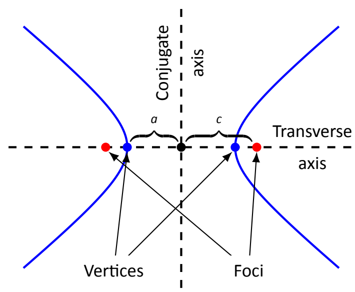


Figure 9.1.12: Labelling the significant features of a hyperbola.

We do not have a convenient way of visualizing the construction of a hyperbola as we did for the ellipse. The geometric definition does allow us to find an algebraic expression that describes it. It will be useful to define some terms first.

The two foci lie on the **transverse axis** of the hyperbola; the midpoint of the line segment joining the foci is the **center** of the hyperbola. The transverse axis intersects the hyperbola at two points, each a **vertex** of the hyperbola. The line through the center and perpendicular to the transverse axis is the **conjugate axis.** This is illustrated in Figure 9.1.12. It is easy to show that the constant difference of distances used in the definition of the hyperbola is the distance between the vertices, i.e., $2a$.

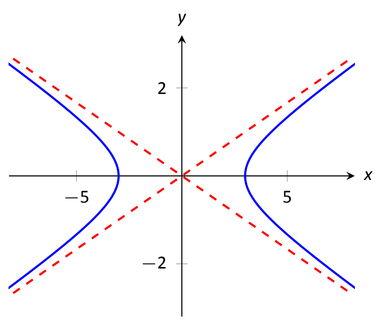


Figure 9.1.13: Graphing the hyperbola $\frac{x^2}{9} - \frac{y^2}{1} = 1$ along with its asymptotes, $y = \pm x/3$.

Key Idea 9.1.3 Standard Equation of a Hyperbola

The equation of a hyperbola centered at (h, k) in standard form is:

1. **Horizontal Transverse Axis:** $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1.$
2. **Vertical Transverse Axis:** $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1.$

The vertices are located a units from the center and the foci are located c units from the center, where $c^2 = a^2 + b^2$.

Graphing Hyperbolas

Consider the hyperbola $\frac{x^2}{9} - \frac{y^2}{1} = 1$. Solving for y, we find $y = \pm\sqrt{x^2/9 - 1}$. As x grows large, the "-1" part of the equation for y becomes less significant and $y \approx \pm\sqrt{x^2/9} = \pm x/3$. That is, as x gets large, the graph of the hyperbola looks very much like the lines $y = \pm x/3$. These lines are asymptotes of the hyperbola, as shown in Figure 9.1.13.

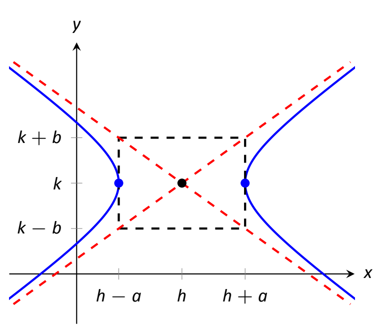


Figure 9.1.14: Using the asymptotes of a hyperbola as a graphing aid.

This is a valuable tool in sketching. Given the equation of a hyperbola in general form, draw a rectangle centered at (h, k) with sides of length $2a$ parallel to the transverse axis and sides of length $2b$ parallel to the conjugate axis. (See Figure 9.1.14 for an example with a horizontal transverse axis.) The diagonals of the rectangle lie on the asymptotes.

These lines pass through (h, k). When the transverse axis is horizontal, the slopes are $\pm b/a$; when the transverse axis is vertical, their slopes are $\pm a/b$. This gives equations:

Horizontal Transverse Axis	Vertical Transverse Axis
$y = \pm\frac{b}{a}(x-h)+k$	$y = \pm\frac{a}{b}(x-h)+k.$

Example 9.1.5 Graphing a hyperbola

Sketch the hyperbola given by $\frac{(y-2)^2}{25} - \frac{(x-1)^2}{4} = 1.$

Solution The hyperbola is centred at $(1,2)$; $a = 5$ and $b = 2$. In Figure 9.1.15 we draw the prescribed rectangle centred at $(1,2)$ along with the asymptotes defined by its diagonals. The hyperbola has a vertical transverse axis, so the vertices are located at $(1,7)$ and $(1,-3)$. This is enough to make a good sketch.

We also find the location of the foci: as $c^2 = a^2 + b^2$, we have $c = \sqrt{29} \approx 5.4$. Thus the foci are located at $(1, 2 \pm 5.4)$ as shown in the figure.

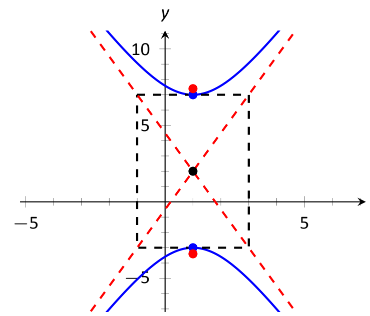


Figure 9.1.15: Graphing the hyperbola in Example 9.1.5.

Example 9.1.6 Graphing a hyperbola

Sketch the hyperbola given by $9x^2 - y^2 + 2y = 10$.

Solution We must complete the square to put the equation in general form. (We recognize this as a hyperbola since it is a general quadratic equation and the x^2 and y^2 terms have opposite signs.)

$$\begin{aligned} 9x^2 - y^2 + 2y &= 10 \\ 9x^2 - (y^2 - 2y) &= 10 \\ 9x^2 - (y^2 - 2y + 1 - 1) &= 10 \\ 9x^2 - \left((y-1)^2 - 1\right) &= 10 \\ 9x^2 - (y-1)^2 &= 9 \\ x^2 - \frac{(y-1)^2}{9} &= 1 \end{aligned}$$

We see the hyperbola is centred at $(0,1)$, with a horizontal transverse axis, where $a = 1$ and $b = 3$. The appropriate rectangle is sketched in Figure 9.1.16 along with the asymptotes of the hyperbola. The vertices are located at $(\pm 1, 1)$. We have $c = \sqrt{10} \approx 3.2$, so the foci are located at $(\pm 3.2, 1)$ as shown in the figure.

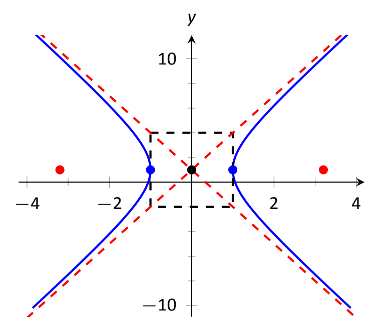


Figure 9.1.16: Graphing the hyperbola in Example 9.1.6.

Eccentricity

Definition 9.1.5 Eccentricity of a Hyperbola

The eccentricity of a hyperbola is $e = \dfrac{c}{a}$.

Note that this is the definition of eccentricity as used for the ellipse. When c is close in value to a (i.e., $e \approx 1$), the hyperbola is very narrow (looking almost like crossed lines). Figure 9.1.17 shows hyperbolas centered at the origin with $a = 1$. The graph in (a) has $c = 1.05$, giving an eccentricity of $e = 1.05$, which is close to 1. As c grows larger, the hyperbola widens and begins to look like parallel lines, as shown in part (d) of the figure.

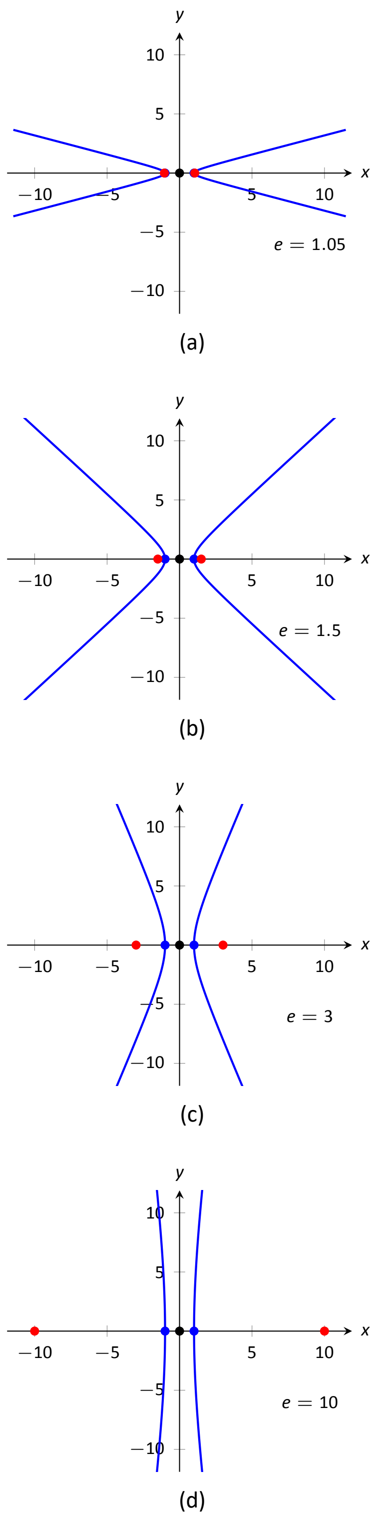


Figure 9.1.17: Understanding the eccentricity of a hyperbola.

Reflective Property

Hyperbolas share a similar reflective property with ellipses. However, in the case of a hyperbola, a ray emanating from a focus that intersects the hyperbola reflects along a line containing the other focus, but moving *away* from that focus. This is illustrated in Figure 9.1.19 (on the next page). Hyperbolic mirrors are commonly used in telescopes because of this reflective property. It is stated formally in the following theorem.

Theorem 9.1.3 Reflective Property of Hyperbolas

Let P be a point on a hyperbola with foci F_1 and F_2. The tangent line to the hyperbola at P makes equal angles with the following two lines:

1. The line through F_1 and P, and
2. The line through F_2 and P.

Location Determination

Determining the location of a known event has many practical uses (locating the epicenter of an earthquake, an airplane crash site, the position of the person speaking in a large room, etc.).

To determine the location of an earthquake's epicenter, seismologists use *trilateration* (not to be confused with *triangulation*). A seismograph allows one to determine how far away the epicenter was; using three separate readings, the location of the epicenter can be approximated.

A key to this method is knowing distances. What if this information is not available? Consider three microphones at positions A, B and C which all record a noise (a person's voice, an explosion, etc.) created at unknown location D. The microphone does not "know" when the sound was *created*, only when the sound was *detected*. How can the location be determined in such a situation?

If each location has a clock set to the same time, hyperbolas can be used to determine the location. Suppose the microphone at position A records the sound at exactly 12:00, location B records the time exactly 1 second later, and location C records the noise exactly 2 seconds after that. We are interested in the *difference* of times. Since the speed of sound is approximately 340 m/s, we can conclude quickly that the sound was created 340 meters closer to position A than position B. If A and B are a known distance apart (as shown in Figure 9.1.18 (a)), then we can determine a hyperbola on which D must lie.

The "difference of distances" is 340; this is also the distance between vertices of the hyperbola. So we know $2a = 340$. Positions A and B lie on the foci, so $2c = 1000$. From this we can find $b \approx 470$ and can sketch the hyperbola, given in part (b) of the figure. We only care about the side closest to A. (Why?)

We can also find the hyperbola defined by positions B and C. In this case, $2a = 680$ as the sound travelled an extra 2 seconds to get to C. We still have $2c = 1000$, centring this hyperbola at $(-500, 500)$. We find $b \approx 367$. This hyperbola is sketched in part (c) of the figure. The intersection point of the two graphs is the location of the sound, at approximately $(188, -222.5)$.

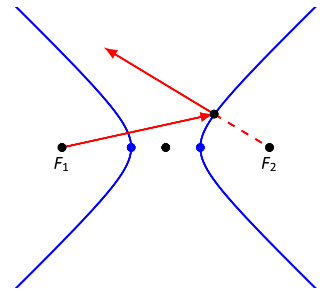


Figure 9.1.19: Illustrating the reflective property of a hyperbola.

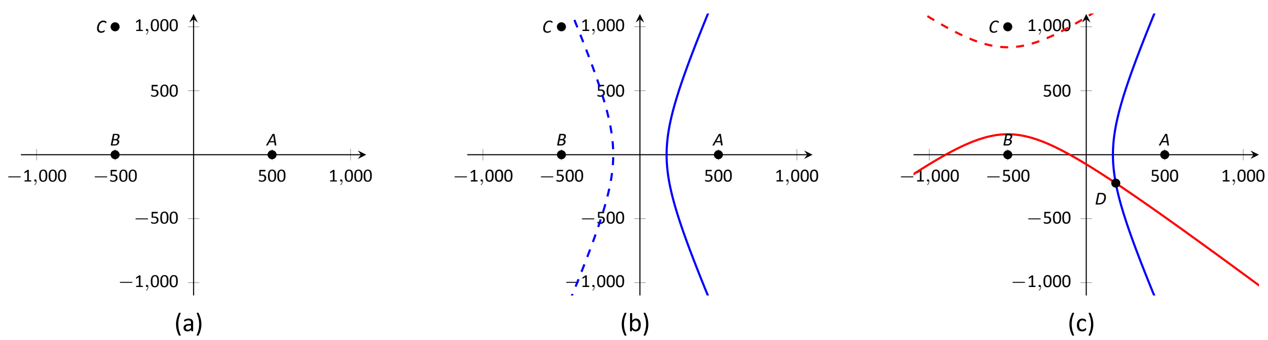


Figure 9.1.18: Using hyperbolas in location detection.

This chapter explores curves in the plane, in particular curves that cannot be described by functions of the form $y = f(x)$. In this section, we learned of ellipses and hyperbolas that are defined implicitly, not explicitly. In the following sections, we will learn completely new ways of describing curves in the plane, using *parametric equations* and *polar coordinates*, then study these curves using calculus techniques.

Exercises 9.1

Terms and Concepts

1. What is the difference between degenerate and nondegenerate conics?

2. Use your own words to explain what the eccentricity of an ellipse measures.

3. What has the largest eccentricity: an ellipse or a hyperbola?

4. Explain why the following is true: "If the coefficient of the x^2 term in the equation of an ellipse in standard form is smaller than the coefficient of the y^2 term, then the ellipse has a horizontal major axis."

5. Explain how one can quickly look at the equation of a hyperbola in standard form and determine whether the transverse axis is horizontal or vertical.

6. Fill in the blank: It can be said that ellipses and hyperbolas share the *same* reflective property: "A ray emanating from one focus will reflect off the conic along a ________ that contains the other focus."

Problems

In Exercises 7 – 14, find the equation of the parabola defined by the given information. Sketch the parabola.

7. Focus: $(3, 2)$; directrix: $y = 1$

8. Focus: $(-1, -4)$; directrix: $y = 2$

9. Focus: $(1, 5)$; directrix: $x = 3$

10. Focus: $(1/4, 0)$; directrix: $x = -1/4$

11. Focus: $(1, 1)$; vertex: $(1, 2)$

12. Focus: $(-3, 0)$; vertex: $(0, 0)$

13. Vertex: $(0, 0)$; directrix: $y = -1/16$

14. Vertex: $(2, 3)$; directrix: $x = 4$

In Exercises 15 – 16, the equation of a parabola and a point on its graph are given. Find the focus and directrix of the parabola, and verify that the given point is equidistant from the focus and directrix.

15. $y = \frac{1}{4}x^2$, $P = (2, 1)$

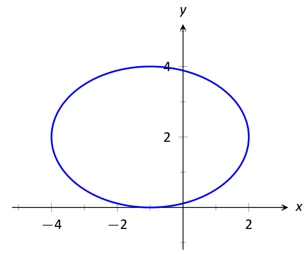
16. $x = \frac{1}{8}(y-2)^2 + 3$, $P = (11, 10)$

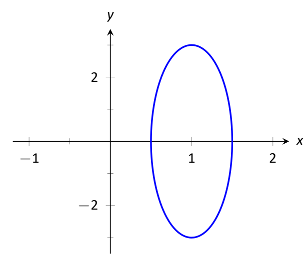
In Exercises 17 – 18, sketch the ellipse defined by the given equation. Label the center, foci and vertices.

17. $\dfrac{(x-1)^2}{3} + \dfrac{(y-2)^2}{5} = 1$

18. $\dfrac{1}{25}x^2 + \dfrac{1}{9}(y+3)^2 = 1$

In Exercises 19 – 20, find the equation of the ellipse shown in the graph. Give the location of the foci and the eccentricity of the ellipse.

19. 

20. 

In Exercises 21 – 24, find the equation of the ellipse defined by the given information. Sketch the elllipse.

21. Foci: $(\pm 2, 0)$; vertices: $(\pm 3, 0)$

22. Foci: $(-1, 3)$ and $(5, 3)$; vertices: $(-3, 3)$ and $(7, 3)$

23. Foci: $(2, \pm 2)$; vertices: $(2, \pm 7)$

24. Focus: $(-1, 5)$; vertex: $(-1, -4)$; center: $(-1, 1)$

In Exercises 25 – 28, write the equation of the given ellipse in standard form.

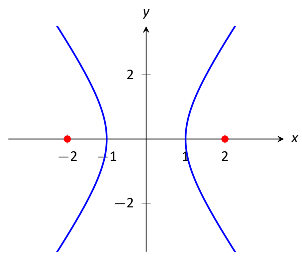
25. $x^2 - 2x + 2y^2 - 8y = -7$

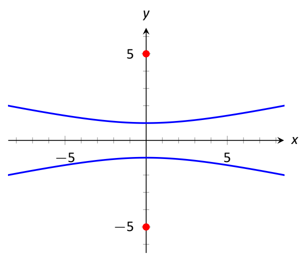
26. $5x^2 + 3y^2 = 15$

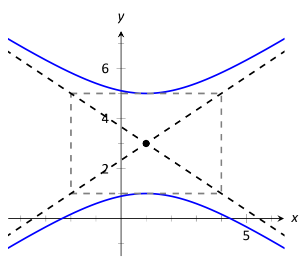
27. $3x^2 + 2y^2 - 12y + 6 = 0$

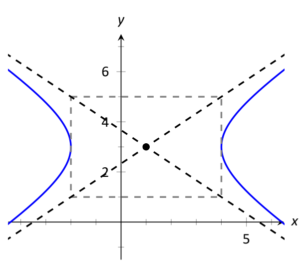
28. $x^2 + y^2 - 4x - 4y + 4 = 0$

In Exercises 29 – 32, find the equation of the hyperbola shown in the graph.

29. 

30. 

31. 

32. 

In Exercises 33 – 34, sketch the hyperbola defined by the given equation. Label the center and foci.

33. $\dfrac{(x-1)^2}{16} - \dfrac{(y+2)^2}{9} = 1$

34. $(y-4)^2 - \dfrac{(x+1)^2}{25} = 1$

In Exercises 35 – 38, find the equation of the hyperbola defined by the given information. Sketch the hyperbola.

35. Foci: $(\pm 3, 0)$; vertices: $(\pm 2, 0)$

36. Foci: $(0, \pm 3)$; vertices: $(0, \pm 2)$

37. Foci: $(-2, 3)$ and $(8, 3)$; vertices: $(-1, 3)$ and $(7, 3)$

38. Foci: $(3, -2)$ and $(3, 8)$; vertices: $(3, 0)$ and $(3, 6)$

In Exercises 39 – 42, write the equation of the hyperbola in standard form.

39. $3x^2 - 4y^2 = 12$

40. $3x^2 - y^2 + 2y = 10$

41. $x^2 - 10y^2 + 40y = 30$

42. $(4y - x)(4y + x) = 4$

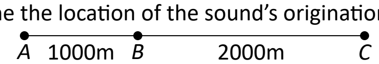
43. Consider the ellipse given by $\dfrac{(x-1)^2}{4} + \dfrac{(y-3)^2}{12} = 1$.
    (a) Verify that the foci are located at $(1, 3 \pm 2\sqrt{2})$.
    (b) The points $P_1 = (2, 6)$ and $P_2 = (1+\sqrt{2}, 3+\sqrt{6}) \approx (2.414, 5.449)$ lie on the ellipse. Verify that the sum of distances from each point to the foci is the same.

44. Johannes Kepler discovered that the planets of our solar system have elliptical orbits with the Sun at one focus. The Earth's elliptical orbit is used as a standard unit of distance; the distance from the center of Earth's elliptical orbit to one vertex is 1 Astronomical Unit, or A.U.
    The following table gives information about the orbits of three planets.

	Distance from center to vertex	eccentricity
Mercury	0.387 A.U.	0.2056
Earth	1 A.U.	0.0167
Mars	1.524 A.U.	0.0934

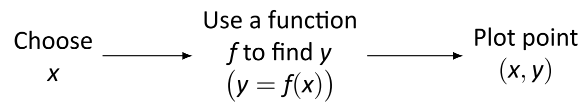
    (a) In an ellipse, knowing $c^2 = a^2 - b^2$ and $e = c/a$ allows us to find b in terms of a and e. Show $b = a\sqrt{1-e^2}$.
    (b) For each planet, find equations of their elliptical orbit of the form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. (This places the center at $(0, 0)$, but the Sun is in a different location for each planet.)
    (c) Shift the equations so that the Sun lies at the origin. Plot the three elliptical orbits.

45. A loud sound is recorded at three stations that lie on a line as shown in the figure below. Station A recorded the sound 1 second after Station B, and Station C recorded the sound 3 seconds after B. Using the speed of sound as 340m/s, determine the location of the sound's origination.



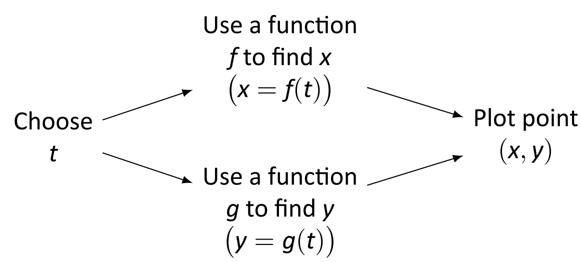
9.2 Parametric Equations

We are familiar with sketching shapes, such as parabolas, by following this basic procedure:



The **rectangular equation** $y = f(x)$ works well for some shapes like a parabola with a vertical axis of symmetry, but in the previous section we encountered several shapes that could not be sketched in this manner. (To plot an ellipse using the above procedure, we need to plot the "top" and "bottom" separately.)

In this section we introduce a new sketching procedure:



Here, x and y are found separately but then plotted together. This leads us to a definition.

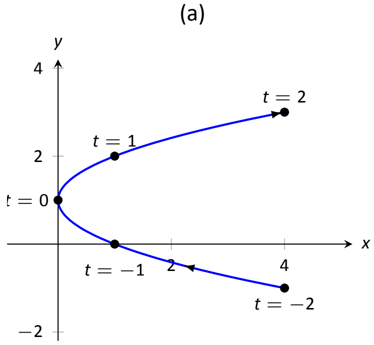
Definition 9.2.1 Parametric Equations and Curves

Let f and g be continuous functions on an interval I. The set of all points $\big(x, y\big) = \big(f(t), g(t)\big)$ in the Cartesian plane, as t varies over I, is the **graph** of the **parametric equations** $x = f(t)$ and $y = g(t)$, where t is the **parameter**. A **curve** is a graph along with the parametric equations that define it.

This is a formal definition of the word *curve*. When a curve lies in a plane (such as the Cartesian plane), it is often referred to as a **plane curve**. Examples will help us understand the concepts introduced in the definition.

t	x	y
-2	4	-1
-1	1	0
0	0	1
1	1	2
2	4	3

(a)



(b)

Figure 9.2.1: A table of values of the parametric equations in Example 9.2.1 along with a sketch of their graph.

Example 9.2.1 Plotting parametric functions

Plot the graph of the parametric equations $x = t^2$, $y = t + 1$ for t in $[-2, 2]$.

Solution We plot the graphs of parametric equations in much the same manner as we plotted graphs of functions like $y = f(x)$: we make a table of values, plot points, then connect these points with a "reasonable" looking curve. Figure 9.2.1(a) shows such a table of values; note how we have 3 columns.

The points (x, y) from the table are plotted in Figure 9.2.1(b). The points have been connected with a smooth curve. Each point has been labelled with its corresponding t-value. These values, along with the two arrows along the curve, are used to indicate the **orientation** of the graph. This information helps us determine the direction in which the graph is "moving."

We often use the letter t as the parameter as we often regard t as representing *time*. Certainly there are many contexts in which the parameter is not time, but it can be helpful to think in terms of time as one makes sense of parametric plots and their orientation (for instance, "At time $t = 0$ the position is $(1, 2)$ and at time $t = 3$ the position is $(5, 1)$.").

Example 9.2.2 Plotting parametric functions

Sketch the graph of the parametric equations $x = \cos^2 t$, $y = \cos t + 1$ for t in $[0, \pi]$.

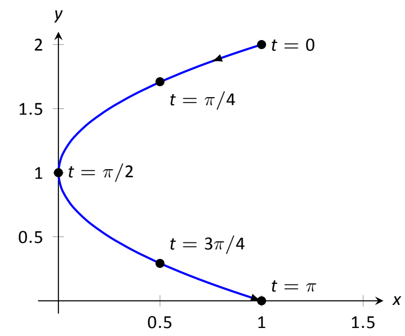
Solution We again start by making a table of values in Figure 9.2.2(a), then plot the points (x, y) on the Cartesian plane in Figure 9.2.2(b).

It is not difficult to show that the curves in Examples 9.2.1 and 9.2.2 are portions of the same parabola. While the *parabola* is the same, the *curves* are different. In Example 9.2.1, if we let t vary over all real numbers, we'd obtain the entire parabola. In this example, letting t vary over all real numbers would still produce the same graph; this portion of the parabola would be traced, and re–traced, infinitely many times. The orientation shown in Figure 9.2.2 shows the orientation on $[0, \pi]$, but this orientation is reversed on $[\pi, 2\pi]$.

These examples begin to illustrate the powerful nature of parametric equations. Their graphs are far more diverse than the graphs of functions produced by "$y = f(x)$" functions.

t	x	y
0	1	2
$\pi/4$	$1/2$	$1 + \sqrt{2}/2$
$\pi/2$	0	1
$3\pi/4$	$1/2$	$1 - \sqrt{2}/2$
π	1	0

(a)



(b)

Figure 9.2.2: A table of values of the parametric equations in Example 9.2.2 along with a sketch of their graph.

Technology Note: Most graphing utilities can graph functions given in parametric form. Often the word "parametric" is abbreviated as "PAR" or "PARAM" in the options. The user usually needs to determine the graphing window (i.e, the minimum and maximum x- and y-values), along with the values of t that are to be plotted. The user is often prompted to give a t minimum, a t maximum, and a "t-step" or "Δt." Graphing utilities effectively plot parametric functions just as we've shown here: they plots lots of points. A smaller t-step plots more points, making for a smoother graph (but may take longer). In Figure 9.2.1, the t-step is 1; in Figure 9.2.2, the t-step is $\pi/4$.

One nice feature of parametric equations is that their graphs are easy to shift. While this is not too difficult in the "$y = f(x)$" context, the resulting function can look rather messy. (Plus, to shift to the right by two, we replace x with $x - 2$, which is counter–intuitive.) The following example demonstrates this.

Example 9.2.3 Shifting the graph of parametric functions

Sketch the graph of the parametric equations $x = t^2 + t$, $y = t^2 - t$. Find new parametric equations that shift this graph to the right 3 places and down 2.

Solution The graph of the parametric equations is given in Figure 9.2.3 (a). It is a parabola with a axis of symmetry along the line $y = x$; the vertex is at $(0, 0)$.

In order to shift the graph to the right 3 units, we need to increase the x-value by 3 for every point. The straightforward way to accomplish this is simply to add 3 to the function defining x: $x = t^2 + t + 3$. To shift the graph down by 2 units, we wish to decrease each y-value by 2, so we subtract 2 from the function defining y: $y = t^2 - t - 2$. Thus our parametric equations for the shifted graph are $x = t^2 + t + 3$, $y = t^2 - t - 2$. This is graphed in Figure 9.2.3 (b). Notice how the vertex is now at $(3, -2)$.

Because the x- and y-values of a graph are determined independently, the

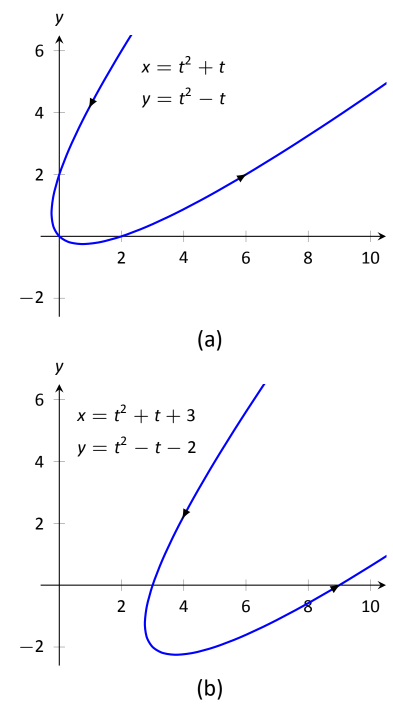


Figure 9.2.3: Illustrating how to shift graphs in Example 9.2.3.

graphs of parametric functions often possess features not seen on "$y = f(x)$" type graphs. The next example demonstrates how such graphs can arrive at the same point more than once.

Example 9.2.4 Graphs that cross themselves

Plot the parametric functions $x = t^3 - 5t^2 + 3t + 11$ and $y = t^2 - 2t + 3$ and determine the t-values where the graph crosses itself.

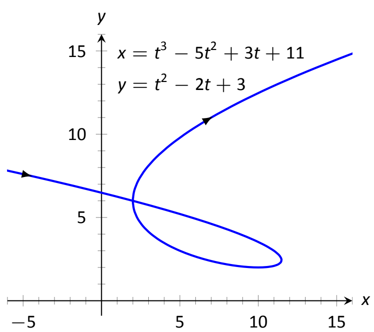


Figure 9.2.4: A graph of the parametric equations from Example 9.2.4.

Solution Using the methods developed in this section, we again plot points and graph the parametric equations as shown in Figure 9.2.4. It appears that the graph crosses itself at the point $(2, 6)$, but we'll need to analytically determine this.

We are looking for two different values, say, s and t, where $x(s) = x(t)$ and $y(s) = y(t)$. That is, the x-values are the same precisely when the y-values are the same. This gives us a system of 2 equations with 2 unknowns:

$$s^3 - 5s^2 + 3s + 11 = t^3 - 5t^2 + 3t + 11$$
$$s^2 - 2s + 3 = t^2 - 2t + 3$$

Solving this system is not trivial but involves only algebra. Using the quadratic formula, one can solve for t in the second equation and find that $t = 1 \pm \sqrt{s^2 - 2s + 1}$. This can be substituted into the first equation, revealing that the graph crosses itself at $t = -1$ and $t = 3$. We confirm our result by computing $x(-1) = x(3) = 2$ and $y(-1) = y(3) = 6$.

Converting between rectangular and parametric equations

It is sometimes useful to rewrite equations in rectangular form (i.e., $y = f(x)$) into parametric form, and vice–versa. Converting from rectangular to parametric can be very simple: given $y = f(x)$, the parametric equations $x = t$, $y = f(t)$ produce the same graph. As an example, given $y = x^2$, the parametric equations $x = t$, $y = t^2$ produce the familiar parabola. However, other parametrizations can be used. The following example demonstrates one possible alternative.

Example 9.2.5 Converting from rectangular to parametric

Consider $y = x^2$. Find parametric equations $x = f(t)$, $y = g(t)$ for the parabola where $t = \frac{dy}{dx}$. That is, $t = a$ corresponds to the point on the graph whose tangent line has slope a.

Solution We start by computing $\frac{dy}{dx}$: $y' = 2x$. Thus we set $t = 2x$. We can solve for x and find $x = t/2$. Knowing that $y = x^2$, we have $y = t^2/4$. Thus parametric equations for the parabola $y = x^2$ are

$$x = t/2 \quad y = t^2/4.$$

To find the point where the tangent line has a slope of -2, we set $t = -2$. This gives the point $(-1, 1)$. We can verify that the slope of the line tangent to the curve at this point indeed has a slope of -2.

We sometimes choose the parameter to accurately model physical behaviour.

Example 9.2.6 Converting from rectangular to parametric

An object is fired from a height of 0 feet and lands 6 seconds later, 192 feet away. Assuming ideal projectile motion, the height, in feet, of the object can be described by $h(x) = -x^2/64 + 3x$, where x is the distance in feet from the initial location. (Thus $h(0) = h(192) = 0$ ft.) Find parametric equations $x = f(t)$,

$y = g(t)$ for the path of the projectile where x is the horizontal distance the object has travelled at time t (in seconds) and y is the height at time t.

Solution Physics tells us that the horizontal motion of the projectile is linear; that is, the horizontal speed of the projectile is constant. Since the object travels 192 ft in 6 s, we deduce that the object is moving horizontally at a rate of 32 ft/s, giving the equation $x = 32t$. As $y = -x^2/64 + 3x$, we find $y = -16t^2 + 96t$. We can quickly verify that $y'' = -32$ ft/s^2, the acceleration due to gravity, and that the projectile reaches its maximum at $t = 3$, halfway along its path.

These parametric equations make certain determinations about the object's location easy: 2 seconds into the flight the object is at the point $\big(x(2), y(2)\big) = \big(64, 128\big)$. That is, it has travelled horizontally 64 ft and is at a height of 128 ft, as shown in Figure 9.2.5.

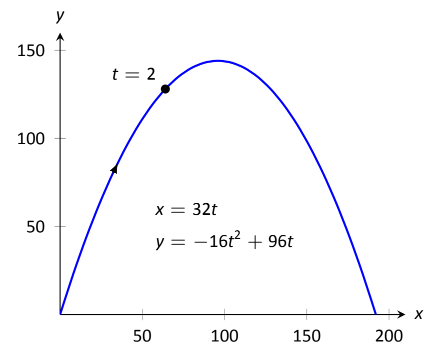


Figure 9.2.5: Graphing projectile motion in Example 9.2.6.

It is sometimes necessary to convert given parametric equations into rectangular form. This can be decidedly more difficult, as some "simple" looking parametric equations can have very "complicated" rectangular equations. This conversion is often referred to as "eliminating the parameter," as we are looking for a relationship between x and y that does not involve the parameter t.

Example 9.2.7 Eliminating the parameter

Find a rectangular equation for the curve described by

$$x = \frac{1}{t^2+1} \quad \text{and} \quad y = \frac{t^2}{t^2+1}.$$

Solution There is not a set way to eliminate a parameter. One method is to solve for t in one equation and then substitute that value in the second. We use that technique here, then show a second, simpler method.

Starting with $x = 1/(t^2+1)$, solve for t: $t = \pm\sqrt{1/x - 1}$. Substitute this value for t in the equation for y:

$$\begin{aligned} y &= \frac{t^2}{t^2+1} \\ &= \frac{1/x - 1}{1/x - 1 + 1} \\ &= \frac{1/x - 1}{1/x} \\ &= \left(\frac{1}{x} - 1\right)\cdot x \\ &= 1 - x. \end{aligned}$$

Thus $y = 1 - x$. One may have recognized this earlier by manipulating the equation for y:

$$y = \frac{t^2}{t^2+1} = 1 - \frac{1}{t^2+1} = 1 - x.$$

This is a shortcut that is very specific to this problem; sometimes shortcuts exist and are worth looking for.

We should be careful to limit the domain of the function $y = 1 - x$. The parametric equations limit x to values in $(0, 1]$, thus to produce the same graph we should limit the domain of $y = 1 - x$ to the same.

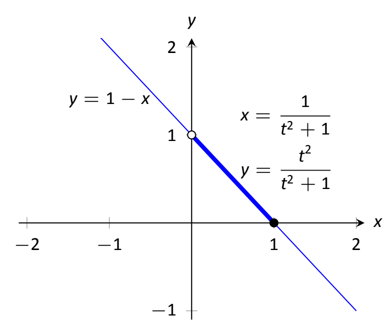


Figure 9.2.6: Graphing parametric and rectangular equations for a graph in Example 9.2.7.

The graphs of these functions is given in Figure 9.2.6. The portion of the graph defined by the parametric equations is given in a thick line; the graph defined by $y = 1 - x$ with unrestricted domain is given in a thin line.

Example 9.2.8 Eliminating the parameter

Eliminate the parameter in $x = 4\cos t + 3$, $y = 2\sin t + 1$

Solution We should not try to solve for t in this situation as the resulting algebra/trig would be messy. Rather, we solve for $\cos t$ and $\sin t$ in each equation, respectively. This gives

$$\cos t = \frac{x-3}{4} \quad \text{and} \quad \sin t = \frac{y-1}{2}.$$

The Pythagorean Theorem gives $\cos^2 t + \sin^2 t = 1$, so:

$$\cos^2 t + \sin^2 t = 1$$
$$\left(\frac{x-3}{4}\right)^2 + \left(\frac{y-1}{2}\right)^2 = 1$$
$$\frac{(x-3)^2}{16} + \frac{(y-1)^2}{4} = 1$$

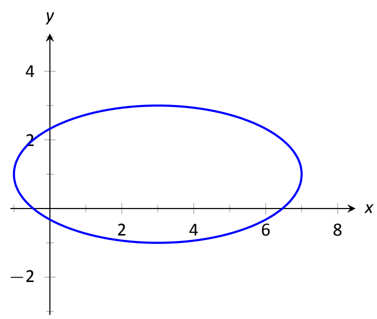


Figure 9.2.7: Graphing the parametric equations $x = 4\cos t + 3$, $y = 2\sin t + 1$ in Example 9.2.8.

This final equation should look familiar – it is the equation of an ellipse! Figure 9.2.7 plots the parametric equations, demonstrating that the graph is indeed of an ellipse with a horizontal major axis and center at $(3, 1)$.

The Pythagorean Theorem can also be used to identify parametric equations for hyperbolas. We give the parametric equations for ellipses and hyperbolas in the following Key Idea.

Key Idea 9.2.1 Parametric Equations of Ellipses and Hyperbolas

- The parametric equations
$$x = a\cos t + h, \quad y = b\sin t + k$$
define an ellipse with horizontal axis of length $2a$ and vertical axis of length $2b$, centred at (h, k).

- The parametric equations
$$x = a\tan t + h, \quad y = \pm b\sec t + k$$
define a hyperbola with vertical transverse axis centred at (h, k), and
$$x = \pm a\sec t + h, \quad y = b\tan t + k$$
defines a hyperbola with horizontal transverse axis. Each has asymptotes at $y = \pm b/a(x - h) + k$.

Special Curves

Figure 9.2.8 gives a small gallery of "interesting" and "famous" curves along with parametric equations that produce them. Interested readers can begin learning more about these curves through internet searches.

One might note a feature shared by two of these graphs: "sharp corners," or **cusps**. We have seen graphs with cusps before and determined that such functions are not differentiable at these points. This leads us to a definition.

Definition 9.2.2 Smooth

A curve C defined by $x = f(t)$, $y = g(t)$ is **smooth** on an interval I if f' and g' are continuous on I and not simultaneously 0 (except possibly at the endpoints of I). A curve is **piecewise smooth** on I if I can be partitioned into subintervals where C is smooth on each subinterval.

Consider the astroid, given by $x = \cos^3 t$, $y = \sin^3 t$. Taking derivatives, we have:

$$x' = -3\cos^2 t \sin t \quad \text{and} \quad y' = 3\sin^2 t \cos t.$$

It is clear that each is 0 when $t = 0,\ \pi/2,\ \pi, \ldots$. Thus the astroid is not smooth at these points, corresponding to the cusps seen in the figure.

We demonstrate this once more.

Example 9.2.9 Determine where a curve is not smooth

Let a curve C be defined by the parametric equations $x = t^3 - 12t + 17$ and $y = t^2 - 4t + 8$. Determine the points, if any, where it is not smooth.

SOLUTION We begin by taking derivatives.

$$x' = 3t^2 - 12, \quad y' = 2t - 4.$$

We set each equal to 0:

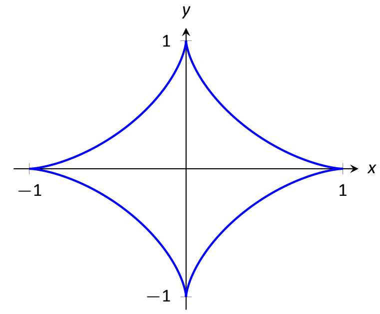
$$x' = 0 \Rightarrow 3t^2 - 12 = 0 \Rightarrow t = \pm 2$$
$$y' = 0 \Rightarrow 2t - 4 = 0 \Rightarrow t = 2$$

We see at $t = 2$ both x' and y' are 0; thus C is not smooth at $t = 2$, corresponding to the point $(1, 4)$. The curve is graphed in Figure 9.2.9, illustrating the cusp at $(1, 4)$.

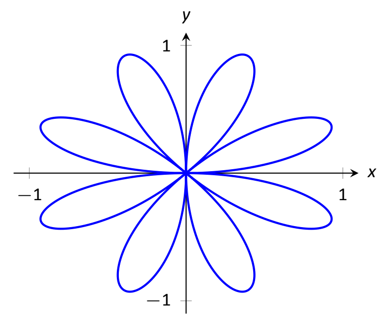
If a curve is not smooth at $t = t_0$, it means that $x'(t_0) = y'(t_0) = 0$ as defined. This, in turn, means that rate of change of x (and y) is 0; that is, at that instant, neither x nor y is changing. If the parametric equations describe the path of some object, this means the object is at rest at t_0. An object at rest can make a "sharp" change in direction, whereas moving objects tend to change direction in a "smooth" fashion.

One should be careful to note that a "sharp corner" does not have to occur when a curve is not smooth. For instance, one can verify that $x = t^3$, $y = t^6$ produce the familiar $y = x^2$ parabola. However, in this parametrization, the curve is not smooth. A particle travelling along the parabola according to the given parametric equations comes to rest at $t = 0$, though no sharp point is created.

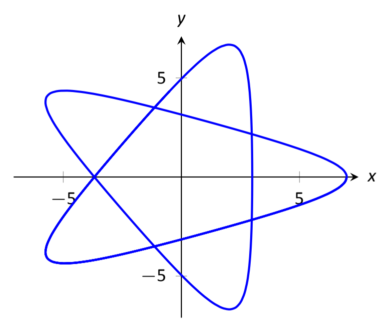
Our previous experience with cusps taught us that a function was not differentiable at a cusp. This can lead us to wonder about derivatives in the context



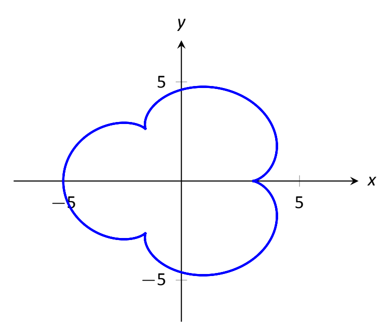
Astroid
$x = \cos^3 t$
$y = \sin^3 t$



Rose Curve
$x = \cos(t)\sin(4t)$
$y = \sin(t)\sin(4t)$



Hypotrochoid
$x = 2\cos(t) + 5\cos(2t/3)$
$y = 2\sin(t) - 5\sin(2t/3)$



Epicycloid
$x = 4\cos(t) - \cos(4t)$
$y = 4\sin(t) - \sin(4t)$

Figure 9.2.8: A gallery of interesting planar curves.

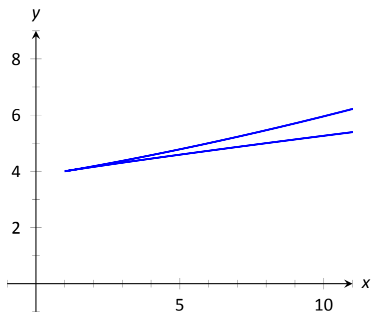


Figure 9.2.9: Graphing the curve in Example 9.2.9; note it is not smooth at $(1,4)$.

of parametric equations and the application of other calculus concepts. Given a curve defined parametrically, how do we find the slopes of tangent lines? Can we determine concavity? We explore these concepts and more in the next section.

Exercises 9.2

Terms and Concepts

1. T/F: When sketching the graph of parametric equations, the x and y values are found separately, then plotted together.

2. The direction in which a graph is "moving" is called the ____ of the graph.

3. An equation written as $y = f(x)$ is written in ____ form.

4. Create parametric equations $x = f(t)$, $y = g(t)$ and sketch their graph. Explain any interesting features of your graph based on the functions f and g.

Problems

In Exercises 5 – 8, sketch the graph of the given parametric equations by hand, making a table of points to plot. Be sure to indicate the orientation of the graph.

5. $x = t^2 + t, \quad y = 1 - t^2, \quad -3 \le t \le 3$

6. $x = 1, \quad y = 5\sin t, \quad -\pi/2 \le t \le \pi/2$

7. $x = t^2, \quad y = 2, \quad -2 \le t \le 2$

8. $x = t^3 - t + 3, \quad y = t^2 + 1, \quad -2 \le t \le 2$

In Exercises 9 – 18, sketch the graph of the given parametric equations; using a graphing utility is advisable. Be sure to indicate the orientation of the graph.

9. $x = t^3 - 2t^2, \quad y = t^2, \quad -2 \le t \le 3$

10. $x = 1/t, \quad y = \sin t, \quad 0 < t \le 10$

11. $x = 3\cos t, \quad y = 5\sin t, \quad 0 \le t \le 2\pi$

12. $x = 3\cos t + 2, \quad y = 5\sin t + 3, \quad 0 \le t \le 2\pi$

13. $x = \cos t, \quad y = \cos(2t), \quad 0 \le t \le \pi$

14. $x = \cos t, \quad y = \sin(2t), \quad 0 \le t \le 2\pi$

15. $x = 2\sec t, \quad y = 3\tan t, \quad -\pi/2 < t < \pi/2$

16. $x = \cosh t, \quad y = \sinh t, \quad -2 \le t \le 2$

17. $x = \cos t + \frac{1}{4}\cos(8t), \quad y = \sin t + \frac{1}{4}\sin(8t), \quad 0 \le t \le 2\pi$

18. $x = \cos t + \frac{1}{4}\sin(8t), \quad y = \sin t + \frac{1}{4}\cos(8t), \quad 0 \le t \le 2\pi$

In Exercises 19 – 20, four sets of parametric equations are given. Describe how their graphs are similar and different. Be sure to discuss orientation and ranges.

19. (a) $x = t \quad y = t^2, \quad -\infty < t < \infty$

    (b) $x = \sin t \quad y = \sin^2 t, \quad -\infty < t < \infty$

    (c) $x = e^t \quad y = e^{2t}, \quad -\infty < t < \infty$

    (d) $x = -t \quad y = t^2, \quad -\infty < t < \infty$

20. (a) $x = \cos t \quad y = \sin t, \quad 0 \le t \le 2\pi$

    (b) $x = \cos(t^2) \quad y = \sin(t^2), \quad 0 \le t \le 2\pi$

    (c) $x = \cos(1/t) \quad y = \sin(1/t), \quad 0 < t < 1$

    (d) $x = \cos(\cos t) \quad y = \sin(\cos t), \quad 0 \le t \le 2\pi$

In Exercises 21 – 30, eliminate the parameter in the given parametric equations.

21. $x = 2t + 5, \quad y = -3t + 1$

22. $x = \sec t, \quad y = \tan t$

23. $x = 4\sin t + 1, \quad y = 3\cos t - 2$

24. $x = t^2, \quad y = t^3$

25. $x = \dfrac{1}{t+1}, \quad y = \dfrac{3t+5}{t+1}$

26. $x = e^t, \quad y = e^{3t} - 3$

27. $x = \ln t, \quad y = t^2 - 1$

28. $x = \cot t, \quad y = \csc t$

29. $x = \cosh t, \quad y = \sinh t$

30. $x = \cos(2t), \quad y = \sin t$

In Exercises 31 – 34, eliminate the parameter in the given parametric equations. Describe the curve defined by the parametric equations based on its rectangular form.

31. $x = at + x_0, \quad y = bt + y_0$

32. $x = r\cos t, \quad y = r\sin t$

33. $x = a\cos t + h, \quad y = b\sin t + k$

34. $x = a\sec t + h, \quad y = b\tan t + k$

In Exercises 35 – 38, find parametric equations for the given rectangular equation using the parameter $t = \frac{dy}{dx}$. Verify that at $t = 1$, the point on the graph has a tangent line with slope of 1.

35. $y = 3x^2 - 11x + 2$

36. $y = e^x$

37. $y = \sin x$ on $[0, \pi]$

38. $y = \sqrt{x}$ on $[0, \infty)$

In Exercises 39 – 42, find the values of t where the graph of the parametric equations crosses itself.

39. $x = t^3 - t + 3, \quad y = t^2 - 3$

40. $x = t^3 - 4t^2 + t + 7, \quad y = t^2 - t$

41. $x = \cos t, \quad y = \sin(2t)$ on $[0, 2\pi]$

42. $x = \cos t \cos(3t), \quad y = \sin t \cos(3t)$ on $[0, \pi]$

In Exercises 43 – 46, find the value(s) of t where the curve defined by the parametric equations is not smooth.

43. $x = t^3 + t^2 - t, \quad y = t^2 + 2t + 3$

44. $x = t^2 - 4t, \quad y = t^3 - 2t^2 - 4t$

45. $x = \cos t, \quad y = 2\cos t$

46. $x = 2\cos t - \cos(2t), \quad y = 2\sin t - \sin(2t)$

In Exercises 47 – 55, find parametric equations that describe the given situation.

47. A projectile is fired from a height of 0ft, landing 16ft away in 4s.

48. A projectile is fired from a height of 0ft, landing 200ft away in 4s.

49. A projectile is fired from a height of 0ft, landing 200ft away in 20s.

50. A circle of radius 2, centered at the origin, that is traced clockwise once on $[0, 2\pi]$.

51. A circle of radius 3, centered at $(1, 1)$, that is traced once counter–clockwise on $[0, 1]$.

52. An ellipse centered at $(1, 3)$ with vertical major axis of length 6 and minor axis of length 2.

53. An ellipse with foci at $(\pm 1, 0)$ and vertices at $(\pm 5, 0)$.

54. A hyperbola with foci at $(5, -3)$ and $(-1, -3)$, and with vertices at $(1, -3)$ and $(3, -3)$.

55. A hyperbola with vertices at $(0, \pm 6)$ and asymptotes $y = \pm 3x$.

9.3 Calculus and Parametric Equations

The previous section defined curves based on parametric equations. In this section we'll employ the techniques of calculus to study these curves.

We are still interested in lines tangent to points on a curve. They describe how the y-values are changing with respect to the x-values, they are useful in making approximations, and they indicate instantaneous direction of travel.

The slope of the tangent line is still $\frac{dy}{dx}$, and the Chain Rule allows us to calculate this in the context of parametric equations. If $x = f(t)$ and $y = g(t)$, the Chain Rule states that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{dy}{dt} \Bigg/ \frac{dx}{dt} = \frac{g'(t)}{f'(t)},$$

provided that $f'(t) \neq 0$. This is important so we label it a Key Idea.

Key Idea 9.3.1 Finding $\frac{dy}{dx}$ with Parametric Equations.

Let $x = f(t)$ and $y = g(t)$, where f and g are differentiable on some open interval I and $f'(t) \neq 0$ on I. Then

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}.$$

We use this to define the tangent line.

Definition 9.3.1 Tangent and Normal Lines

Let a curve C be parametrized by $x = f(t)$ and $y = g(t)$, where f and g are differentiable functions on some interval I containing $t = t_0$. The **tangent line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = g'(t_0)/f'(t_0)$, provided $f'(t_0) \neq 0$.

The **normal line** to C at $t = t_0$ is the line through $\big(f(t_0), g(t_0)\big)$ with slope $m = -f'(t_0)/g'(t_0)$, provided $g'(t_0) \neq 0$.

The definition leaves two special cases to consider. When the tangent line is horizontal, the normal line is undefined by the above definition as $g'(t_0) = 0$. Likewise, when the normal line is horizontal, the tangent line is undefined. It seems reasonable that these lines be defined (one can draw a line tangent to the "right side" of a circle, for instance), so we add the following to the above definition.

1. If the tangent line at $t = t_0$ has a slope of 0, the normal line to C at $t = t_0$ is the line $x = f(t_0)$.

2. If the normal line at $t = t_0$ has a slope of 0, the tangent line to C at $t = t_0$ is the line $x = f(t_0)$.

Example 9.3.1 Tangent and Normal Lines to Curves

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$, and let C be the curve defined by these equations.

1. Find the equations of the tangent and normal lines to C at $t = 3$.
2. Find where C has vertical and horizontal tangent lines.

Solution

1. We start by computing $f'(t) = 10t - 6$ and $g'(t) = 2t + 6$. Thus

$$\frac{dy}{dx} = \frac{2t+6}{10t-6}.$$

Make note of something that might seem unusual: $\frac{dy}{dx}$ is a function of t, not x. Just as points on the curve are found in terms of t, so are the slopes of the tangent lines.

The point on C at $t = 3$ is $(31, 26)$. The slope of the tangent line is $m = 1/2$ and the slope of the normal line is $m = -2$. Thus,

- the equation of the tangent line is $y = \frac{1}{2}(x - 31) + 26$, and
- the equation of the normal line is $y = -2(x - 31) + 26$.

This is illustrated in Figure 9.3.1.

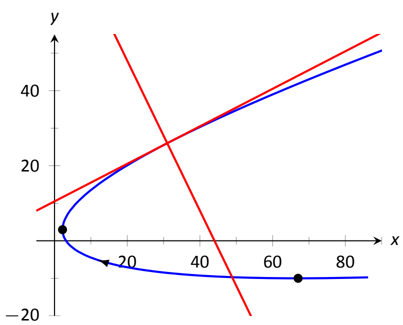


Figure 9.3.1: Graphing tangent and normal lines in Example 9.3.1.

2. To find where C has a horizontal tangent line, we set $\frac{dy}{dx} = 0$ and solve for t. In this case, this amounts to setting $g'(t) = 0$ and solving for t (and making sure that $f'(t) \neq 0$).

$$g'(t) = 0 \quad \Rightarrow \quad 2t + 6 = 0 \quad \Rightarrow \quad t = -3.$$

The point on C corresponding to $t = -3$ is $(67, -10)$; the tangent line at that point is horizontal (hence with equation $y = -10$).

To find where C has a vertical tangent line, we find where it has a horizontal normal line, and set $-\frac{f'(t)}{g'(t)} = 0$. This amounts to setting $f'(t) = 0$ and solving for t (and making sure that $g'(t) \neq 0$).

$$f'(t) = 0 \quad \Rightarrow \quad 10t - 6 = 0 \quad \Rightarrow \quad t = 0.6.$$

The point on C corresponding to $t = 0.6$ is $(2.2, 2.96)$. The tangent line at that point is $x = 2.2$.

The points where the tangent lines are vertical and horizontal are indicated on the graph in Figure 9.3.1.

Example 9.3.2 Tangent and Normal Lines to a Circle

1. Find where the unit circle, defined by $x = \cos t$ and $y = \sin t$ on $[0, 2\pi]$, has vertical and horizontal tangent lines.
2. Find the equation of the normal line at $t = t_0$.

Solution

1. We compute the derivative following Key Idea 9.3.1:

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)} = -\frac{\cos t}{\sin t}.$$

The derivative is 0 when $\cos t = 0$; that is, when $t = \pi/2,\ 3\pi/2$. These are the points $(0, 1)$ and $(0, -1)$ on the circle.

The normal line is horizontal (and hence, the tangent line is vertical) when $\sin t = 0$; that is, when $t = 0,\ \pi,\ 2\pi$, corresponding to the points $(-1, 0)$ and $(0, 1)$ on the circle. These results should make intuitive sense.

2. The slope of the normal line at $t = t_0$ is $m = \dfrac{\sin t_0}{\cos t_0} = \tan t_0$. This normal line goes through the point $(\cos t_0, \sin t_0)$, giving the line

$$\begin{aligned} y &= \frac{\sin t_0}{\cos t_0}(x - \cos t_0) + \sin t_0 \\ &= (\tan t_0)x, \end{aligned}$$

as long as $\cos t_0 \neq 0$. It is an important fact to recognize that the normal lines to a circle pass through its center, as illustrated in Figure 9.3.2. Stated in another way, any line that passes through the center of a circle intersects the circle at right angles.

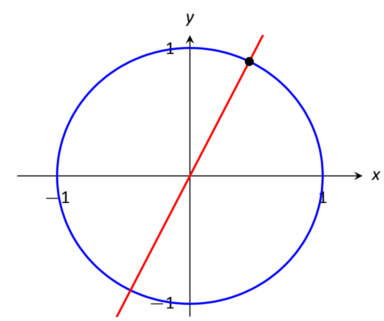


Figure 9.3.2: Illustrating how a circle's normal lines pass through its center.

Example 9.3.3 Tangent lines when $\frac{dy}{dx}$ is not defined

Find the equation of the tangent line to the astroid $x = \cos^3 t$, $y = \sin^3 t$ at $t = 0$, shown in Figure 9.3.3.

Solution We start by finding $x'(t)$ and $y'(t)$:

$$x'(t) = -3\sin t\cos^2 t, \qquad y'(t) = 3\cos t\sin^2 t.$$

Note that both of these are 0 at $t = 0$; the curve is not smooth at $t = 0$ forming a cusp on the graph. Evaluating $\frac{dy}{dx}$ at this point returns the indeterminate form of "0/0".

We can, however, examine the slopes of tangent lines near $t = 0$, and take the limit as $t \to 0$.

$$\begin{aligned} \lim_{t\to 0} \frac{y'(t)}{x'(t)} &= \lim_{t\to 0} \frac{3\cos t\sin^2 t}{-3\sin t\cos^2 t} \qquad \text{(We can cancel as } t \neq 0.\text{)} \\ &= \lim_{t\to 0} -\frac{\sin t}{\cos t} \\ &= 0. \end{aligned}$$

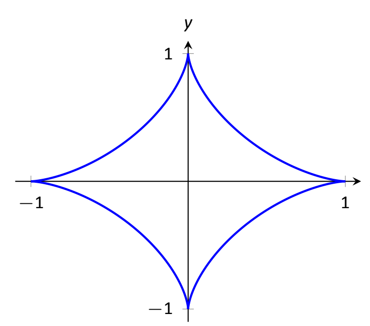


Figure 9.3.3: A graph of an astroid.

We have accomplished something significant. When the derivative $\frac{dy}{dx}$ returns an indeterminate form at $t = t_0$, we can define its value by setting it to be $\displaystyle\lim_{t\to t_0} \frac{dy}{dx}$, if that limit exists. This allows us to find slopes of tangent lines at cusps, which can be very beneficial.

We found the slope of the tangent line at $t = 0$ to be 0; therefore the tangent line is $y = 0$, the x-axis.

Concavity

We continue to analyze curves in the plane by considering their concavity; that is, we are interested in $\frac{d^2y}{dx^2}$, "the second derivative of y with respect to x." To find this, we need to find the derivative of $\frac{dy}{dx}$ with respect to x; that is,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right],$$

but recall that $\frac{dy}{dx}$ is a function of t, not x, making this computation not straightforward.

To make the upcoming notation a bit simpler, let $h(t) = \frac{dy}{dx}$. We want $\frac{d}{dx}[h(t)]$; that is, we want $\frac{dh}{dx}$. We again appeal to the Chain Rule. Note:

$$\frac{dh}{dt} = \frac{dh}{dx}\cdot\frac{dx}{dt} \quad \Rightarrow \quad \frac{dh}{dx} = \frac{dh}{dt}\Big/\frac{dx}{dt}.$$

In words, to find $\frac{d^2y}{dx^2}$, we first take the derivative of $\frac{dy}{dx}$ *with respect to t*, then divide by $x'(t)$. We restate this as a Key Idea.

Key Idea 9.3.2 Finding $\frac{d^2y}{dx^2}$ with Parametric Equations

Let $x = f(t)$ and $y = g(t)$ be twice differentiable functions on an open interval I, where $f'(t) \neq 0$ on I. Then

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\Big/\frac{dx}{dt} = \frac{d}{dt}\left[\frac{dy}{dx}\right]\Big/ f'(t).$$

Examples will help us understand this Key Idea.

Example 9.3.4 Concavity of Plane Curves

Let $x = 5t^2 - 6t + 4$ and $y = t^2 + 6t - 1$ as in Example 9.3.1. Determine the t-intervals on which the graph is concave up/down.

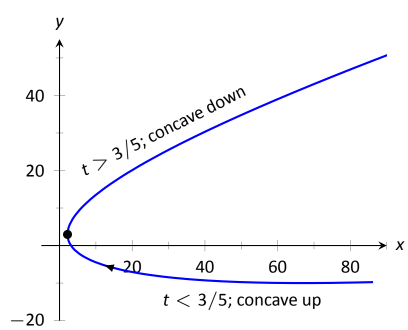


Figure 9.3.4: Graphing the parametric equations in Example 9.3.4 to demonstrate concavity.

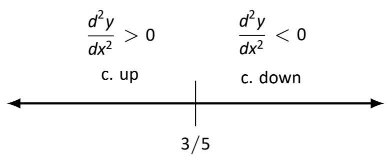
Solution Concavity is determined by the second derivative of y with respect to x, $\frac{d^2y}{dx^2}$, so we compute that here following Key Idea 9.3.2.

In Example 9.3.1, we found $\dfrac{dy}{dx} = \dfrac{2t+6}{10t-6}$ and $f'(t) = 10t - 6$. So:

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dt}\left[\frac{2t+6}{10t-6}\right]\Big/(10t-6)\\ &= -\frac{72}{(10t-6)^2}\Big/(10t-6)\\ &= -\frac{72}{(10t-6)^3}\\ &= -\frac{9}{(5t-3)^3}\end{aligned}$$

The graph of the parametric functions is concave up when $\frac{d^2y}{dx^2} > 0$ and concave down when $\frac{d^2y}{dx^2} < 0$. We determine the intervals when the second derivative is greater/less than 0 by first finding when it is 0 or undefined.

As the numerator of $-\dfrac{9}{(5t-3)^3}$ is never 0, $\frac{d^2y}{dx^2} \neq 0$ for all t. It is undefined when $5t-3=0$; that is, when $t=3/5$. Following the work established in Section 3.4, we look at values of t greater/less than $3/5$ on a number line:



Reviewing Example 9.3.1, we see that when $t=3/5=0.6$, the graph of the parametric equations has a vertical tangent line. This point is also a point of inflection for the graph, illustrated in Figure 9.3.4.

Example 9.3.5 Concavity of Plane Curves

Find the points of inflection of the graph of the parametric equations $x=\sqrt{t}$, $y=\sin t$, for $0 \leq t \leq 16$.

Solution We need to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{y'(t)}{x'(t)} = \frac{\cos t}{1/(2\sqrt{t})} = 2\sqrt{t}\cos t.$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{x'(t)} = \frac{\cos t/\sqrt{t} - 2\sqrt{t}\sin t}{1/(2\sqrt{t})} = 2\cos t - 4t\sin t.$$

The points of inflection are found by setting $\frac{d^2y}{dx^2}=0$. This is not trivial, as equations that mix polynomials and trigonometric functions generally do not have "nice" solutions.

In Figure 9.3.5(a) we see a plot of the second derivative. It shows that it has zeros at approximately $t=0.5,\ 3.5,\ 6.5,\ 9.5,\ 12.5$ and 16. These approximations are not very good, made only by looking at the graph. Newton's Method provides more accurate approximations. Accurate to 2 decimal places, we have:

$$t = 0.65,\ 3.29,\ 6.36,\ 9.48,\ 12.61 \text{ and } 15.74.$$

The corresponding points have been plotted on the graph of the parametric equations in Figure 9.3.5(b). Note how most occur near the x-axis, but not exactly on the axis.

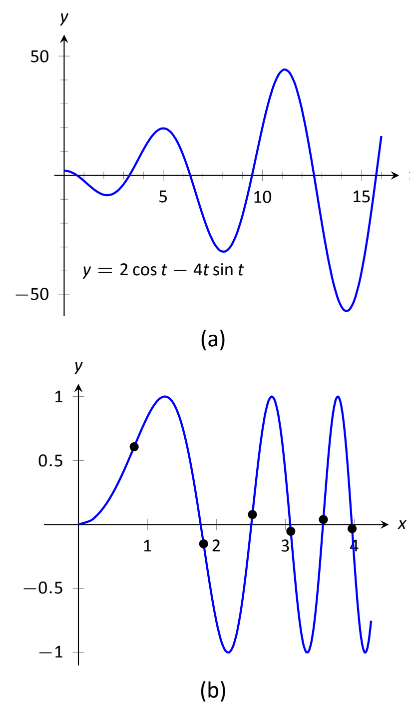


Figure 9.3.5: In (a), a graph of $\frac{d^2y}{dx^2}$, showing where it is approximately 0. In (b), graph of the parametric equations in Example 9.3.5 along with the points of inflection.

Arc Length

We continue our study of the features of the graphs of parametric equations by computing their arc length.

Recall in Section 7.4 we found the arc length of the graph of a function, from $x=a$ to $x=b$, to be

$$L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$$

We can use this equation and convert it to the parametric equation context. Letting $x=f(t)$ and $y=g(t)$, we know that $\frac{dy}{dx}=g'(t)/f'(t)$. It will also be useful to calculate the differential of x:

$$dx = f'(t)dt \qquad \Rightarrow \qquad dt = \frac{1}{f'(t)}\cdot dx.$$

Starting with the arc length formula above, consider:

$$\begin{aligned} L &= \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\ &= \int_a^b \sqrt{1+\frac{g'(t)^2}{f'(t)^2}}\,dx. \end{aligned}$$

Factor out the $f'(t)^2$:

$$\begin{aligned} &= \int_a^b \sqrt{f'(t)^2+g'(t)^2}\cdot \underbrace{\frac{1}{f'(t)}\,dx}_{=dt} \\ &= \int_{t_1}^{t_2} \sqrt{f'(t)^2+g'(t)^2}\,dt. \end{aligned}$$

Note the new bounds (no longer "x" bounds, but "t" bounds). They are found by finding t_1 and t_2 such that $a = f(t_1)$ and $b = f(t_2)$. This formula is important, so we restate it as a theorem.

Theorem 9.3.1 Arc Length of Parametric Curves

Let $x = f(t)$ and $y = g(t)$ be parametric equations with f' and g' continuous on $[t_1, t_2]$, on which the graph traces itself only once. The arc length of the graph, from $t = t_1$ to $t = t_2$, is

$$L = \int_{t_1}^{t_2} \sqrt{f'(t)^2+g'(t)^2}\,dt.$$

Note: Theorem 9.3.1 makes use of differentiability on closed intervals, just as was done in Section 7.4.

As before, these integrals are often not easy to compute. We start with a simple example, then give another where we approximate the solution.

Example 9.3.6 Arc Length of a Circle

Find the arc length of the circle parametrized by $x = 3\cos t$, $y = 3\sin t$ on $[0, 3\pi/2]$.

SOLUTION By direct application of Theorem 9.3.1, we have

$$L = \int_0^{3\pi/2} \sqrt{(-3\sin t)^2 + (3\cos t)^2}\,dt.$$

Apply the Pythagorean Theorem.

$$\begin{aligned} &= \int_0^{3\pi/2} 3\,dt \\ &= 3t\Big|_0^{3\pi/2} = 9\pi/2. \end{aligned}$$

This should make sense; we know from geometry that the circumference of a circle with radius 3 is 6π; since we are finding the arc length of $3/4$ of a circle, the arc length is $3/4 \cdot 6\pi = 9\pi/2$.

Example 9.3.7 Arc Length of a Parametric Curve

The graph of the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ crosses itself as shown in Figure 9.3.6, forming a "teardrop." Find the arc length of the teardrop.

SOLUTION We can see by the parametrizations of x and y that when $t = \pm 1$, $x = 0$ and $y = 0$. This means we'll integrate from $t = -1$ to $t = 1$. Applying Theorem 9.3.1, we have

$$\begin{aligned} L &= \int_{-1}^{1} \sqrt{(3t^2-1)^2 + (2t)^2}\, dt \\ &= \int_{-1}^{1} \sqrt{9t^4 - 2t^2 + 1}\, dt. \end{aligned}$$

Unfortunately, the integrand does not have an antiderivative expressible by elementary functions. We turn to numerical integration to approximate its value. Using 4 subintervals, Simpson's Rule approximates the value of the integral as 2.65051. Using a computer, more subintervals are easy to employ, and $n = 20$ gives a value of 2.71559. Increasing n shows that this value is stable and a good approximation of the actual value.

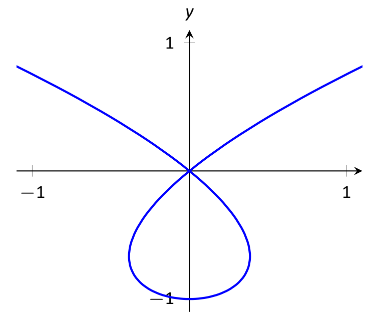


Figure 9.3.6: A graph of the parametric equations in Example 9.3.7, where the arc length of the teardrop is calculated.

Surface Area of a Solid of Revolution

Related to the formula for finding arc length is the formula for finding surface area. We can adapt the formula found in Theorem 7.4.2 from Section 7.4 in a similar way as done to produce the formula for arc length done before.

Theorem 9.3.2 Surface Area of a Solid of Revolution

Consider the graph of the parametric equations $x = f(t)$ and $y = g(t)$, where f' and g' are continuous on an open interval I containing t_1 and t_2 on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the x-axis is (where $g(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} g(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

2. The surface area of the solid formed by revolving the graph about the y-axis is (where $f(t) \geq 0$ on $[t_1, t_2]$):

$$\text{Surface Area} = 2\pi \int_{t_1}^{t_2} f(t)\sqrt{f'(t)^2 + g'(t)^2}\, dt.$$

Example 9.3.8 Surface Area of a Solid of Revolution

Consider the teardrop shape formed by the parametric equations $x = t(t^2 - 1)$, $y = t^2 - 1$ as seen in Example 9.3.7. Find the surface area if this shape is rotated about the x-axis, as shown in Figure 9.3.7.

Solution The teardrop shape is formed between $t = -1$ and $t = 1$. Using Theorem 9.3.2, we see we need for $g(t) \geq 0$ on $[-1, 1]$, and this is not the case. To fix this, we simplify replace $g(t)$ with $-g(t)$, which flips the whole graph about the x-axis (and does not change the surface area of the resulting solid). The surface area is:

$$\begin{aligned}\text{Area } S &= 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{(3t^2 - 1)^2 + (2t)^2}\, dt \\ &= 2\pi \int_{-1}^{1} (1 - t^2)\sqrt{9t^4 - 2t^2 + 1}\, dt.\end{aligned}$$

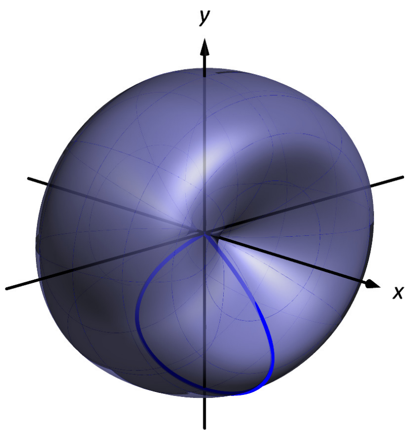


Figure 9.3.7: Rotating a teardrop shape about the x-axis in Example 9.3.8.

Once again we arrive at an integral that we cannot compute in terms of elementary functions. Using Simpson's Rule with $n = 20$, we find the area to be $S = 9.44$. Using larger values of n shows this is accurate to 2 places after the decimal.

After defining a new way of creating curves in the plane, in this section we have applied calculus techniques to the parametric equation defining these curves to study their properties. In the next section, we define another way of forming curves in the plane. To do so, we create a new coordinate system, called *polar coordinates*, that identifies points in the plane in a manner different than from measuring distances from the y- and x- axes.

Exercises 9.3

Terms and Concepts

1. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, $\frac{dy}{dx} = f'(t)/g'(t)$, as long as $g'(t) \neq 0$.

2. Given parametric equations $x = f(t)$ and $y = g(t)$, the derivative $\frac{dy}{dx}$ as given in Key Idea 9.3.1 is a function of ________?

3. T/F: Given parametric equations $x = f(t)$ and $y = g(t)$, to find $\frac{d^2y}{dx^2}$, one simply computes $\frac{d}{dt}\left(\frac{dy}{dx}\right)$.

4. T/F: If $\frac{dy}{dx} = 0$ at $t = t_0$, then the normal line to the curve at $t = t_0$ is a vertical line.

Problems

In Exercises 5 – 12, parametric equations for a curve are given.

(a) Find $\dfrac{dy}{dx}$.

(b) Find the equations of the tangent and normal line(s) at the point(s) given.

(c) Sketch the graph of the parametric functions along with the found tangent and normal lines.

5. $x = t, y = t^2; \quad t = 1$

6. $x = \sqrt{t}, y = 5t + 2; \quad t = 4$

7. $x = t^2 - t, y = t^2 + t; \quad t = 1$

8. $x = t^2 - 1, y = t^3 - t; \quad t = 0$ and $t = 1$

9. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2); \quad t = \pi/4$

10. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]; \quad t = \pi/4$

11. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]; \quad t = 3\pi/4$

12. $x = e^{t/10} \cos t, y = e^{t/10} \sin t; \quad t = \pi/2$

In Exercises 13 – 20, find t-values where the curve defined by the given parametric equations has a horizontal tangent line. Note: these are the same equations as in Exercises 5 – 12.

13. $x = t, y = t^2$

14. $x = \sqrt{t}, y = 5t + 2$

15. $x = t^2 - t, y = t^2 + t$

16. $x = t^2 - 1, y = t^3 - t$

17. $x = \sec t, y = \tan t$ on $(-\pi/2, \pi/2)$

18. $x = \cos t, y = \sin(2t)$ on $[0, 2\pi]$

19. $x = \cos t \sin(2t), y = \sin t \sin(2t)$ on $[0, 2\pi]$

20. $x = e^{t/10} \cos t, y = e^{t/10} \sin t$

In Exercises 21 – 24, find $t = t_0$ where the graph of the given parametric equations is not smooth, then find $\displaystyle\lim_{t \to t_0} \frac{dy}{dx}$.

21. $x = \dfrac{1}{t^2 + 1}, \quad y = t^3$

22. $x = -t^3 + 7t^2 - 16t + 13, \quad y = t^3 - 5t^2 + 8t - 2$

23. $x = t^3 - 3t^2 + 3t - 1, \quad y = t^2 - 2t + 1$

24. $x = \cos^2 t, \quad y = 1 - \sin^2 t$

In Exercises 25 – 32, parametric equations for a curve are given. Find $\frac{d^2y}{dx^2}$, then determine the intervals on which the graph of the curve is concave up/down. Note: these are the same equations as in Exercises 5 – 12.

25. $x = t, \quad y = t^2$

26. $x = \sqrt{t}, \quad y = 5t + 2$

27. $x = t^2 - t, \quad y = t^2 + t$

28. $x = t^2 - 1, \quad y = t^3 - t$

29. $x = \sec t, \quad y = \tan t$ on $(-\pi/2, \pi/2)$

30. $x = \cos t, \quad y = \sin(2t)$ on $[0, 2\pi]$

31. $x = \cos t \sin(2t), \quad y = \sin t \sin(2t)$ on $[-\pi/2, \pi/2]$

32. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$

In Exercises 33 – 36, find the arc length of the graph of the parametric equations on the given interval(s).

33. $x = -3\sin(2t), \quad y = 3\cos(2t)$ on $[0, \pi]$

34. $x = e^{t/10} \cos t, \quad y = e^{t/10} \sin t$ on $[0, 2\pi]$ and $[2\pi, 4\pi]$

35. $x = 5t + 2, \quad y = 1 - 3t$ on $[-1, 1]$

36. $x = 2t^{3/2}, \quad y = 3t$ on $[0, 1]$

In Exercises 37 – 40, numerically approximate the given arc length.

37. Approximate the arc length of one petal of the rose curve $x = \cos t \cos(2t), \quad y = \sin t \cos(2t)$ using Simpson's Rule and $n = 4$.

38. Approximate the arc length of the "bow tie curve" $x = \cos t, \quad y = \sin(2t)$ using Simpson's Rule and $n = 6$.

39. Approximate the arc length of the parabola $x = t^2 - t$, $y = t^2 + t$ on $[-1, 1]$ using Simpson's Rule and $n = 4$.

40. A common approximate of the circumference of an ellipse given by $x = a\cos t, \quad y = b\sin t$ is $C \approx 2\pi\sqrt{\dfrac{a^2 + b^2}{2}}$. Use this formula to approximate the circumference of $x = 5\cos t, \quad y = 3\sin t$ and compare this to the approximation given by Simpson's Rule and $n = 6$.

In Exercises 41 – 44, a solid of revolution is described. Find or approximate its surface area as specified.

41. Find the surface area of the sphere formed by rotating the circle $x = 2\cos t, \quad y = 2\sin t$ about:

    (a) the x-axis and

    (b) the y-axis.

42. Find the surface area of the torus (or "donut") formed by rotating the circle $x = \cos t + 2, \quad y = \sin t$ about the y-axis.

43. Approximate the surface area of the solid formed by rotating the "upper right half" of the bow tie curve $x = \cos t$, $y = \sin(2t)$ on $[0, \pi/2]$ about the x-axis, using Simpson's Rule and $n = 4$.

44. Approximate the surface area of the solid formed by rotating the one petal of the rose curve $x = \cos t\cos(2t)$, $y = \sin t\cos(2t)$ on $[0, \pi/4]$ about the x-axis, using Simpson's Rule and $n = 4$.

9.4 Introduction to Polar Coordinates

We are generally introduced to the idea of graphing curves by relating x-values to y-values through a function f. That is, we set $y = f(x)$, and plot lots of point pairs (x, y) to get a good notion of how the curve looks. This method is useful but has limitations, not least of which is that curves that "fail the vertical line test" cannot be graphed without using multiple functions.

The previous two sections introduced and studied a new way of plotting points in the x, y-plane. Using parametric equations, x and y values are computed independently and then plotted together. This method allows us to graph an extraordinary range of curves. This section introduces yet another way to plot points in the plane: using **polar coordinates**.

Polar Coordinates

Start with a point O in the plane called the **pole** (we will always identify this point with the origin). From the pole, draw a ray, called the **initial ray** (we will always draw this ray horizontally, identifying it with the positive x-axis). A point P in the plane is determined by the distance r that P is from O, and the angle θ formed between the initial ray and the segment $\overline{OP}$ (measured counter-clockwise). We record the distance and angle as an ordered pair (r, θ). To avoid confusion with rectangular coordinates, we will denote polar coordinates with the letter P, as in $P(r, \theta)$. This is illustrated in Figure 9.4.1

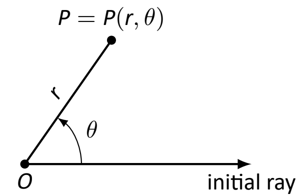


Figure 9.4.1: Illustrating polar coordinates.

Practice will make this process more clear.

Example 9.4.1 Plotting Polar Coordinates

Plot the following polar coordinates:

$$A = P(1, \pi/4) \quad B = P(1.5, \pi) \quad C = P(2, -\pi/3) \quad D = P(-1, \pi/4)$$

SOLUTION To aid in the drawing, a polar grid is provided to the right. To place the point A, go out 1 unit along the initial ray (putting you on the inner circle shown on the grid), then rotate counter-clockwise $\pi/4$ radians (or 45°). Alternately, one can consider the rotation first: think about the ray from O that forms an angle of $\pi/4$ with the initial ray, then move out 1 unit along this ray (again placing you on the inner circle of the grid).

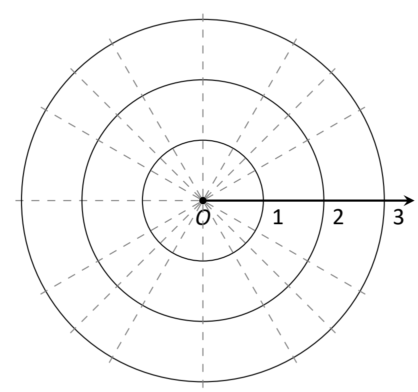


Figure 9.4.2: A polar grid for Example 9.4.1

To plot B, go out 1.5 units along the initial ray and rotate π radians (180°).

To plot C, go out 2 units along the initial ray then rotate *clockwise* $\pi/3$ radians, as the angle given is negative.

To plot D, move along the initial ray "-1" units – in other words, "back up" 1 unit, then rotate counter-clockwise by $\pi/4$. The results are given in Figure 9.4.3.

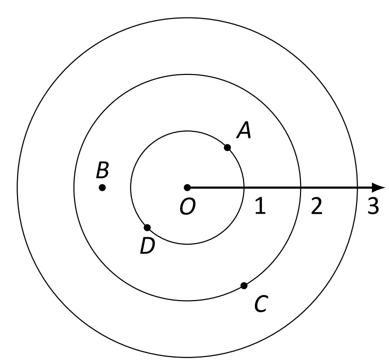


Figure 9.4.3: Plotting polar points in Example 9.4.1.

Consider the following two points: $A = P(1, \pi)$ and $B = P(-1, 0)$. To locate A, go out 1 unit on the initial ray then rotate π radians; to locate B, go out -1 units on the initial ray and don't rotate. One should see that A and B are located at the same point in the plane. We can also consider $C = P(1, 3\pi)$, or $D = P(1, -\pi)$; all four of these points share the same location.

This ability to identify a point in the plane with multiple polar coordinates is both a "blessing" and a "curse." We will see that it is beneficial as we can plot beautiful functions that intersect themselves (much like we saw with parametric functions). The unfortunate part of this is that it can be difficult to determine when this happens. We'll explore this more later in this section.

Polar to Rectangular Conversion

It is useful to recognize both the rectangular (or, Cartesian) coordinates of a point in the plane and its polar coordinates. Figure 9.4.4 shows a point P in the plane with rectangular coordinates (x, y) and polar coordinates $P(r, \theta)$. Using trigonometry, we can make the identities given in the following Key Idea.

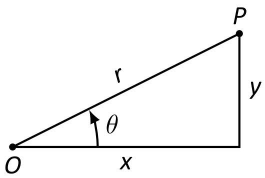


Figure 9.4.4: Converting between rectangular and polar coordinates.

Key Idea 9.4.1 Converting Between Rectangular and Polar Coordinates

Given the polar point $P(r, \theta)$, the rectangular coordinates are determined by

$$x = r\cos\theta \qquad y = r\sin\theta.$$

Given the rectangular coordinates (x, y), the polar coordinates are determined by

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}.$$

Example 9.4.2 Converting Between Polar and Rectangular Coordinates

1. Convert the polar coordinates $P(2, 2\pi/3)$ and $P(-1, 5\pi/4)$ to rectangular coordinates.
2. Convert the rectangular coordinates $(1, 2)$ and $(-1, 1)$ to polar coordinates.

Solution

1. (a) We start with $P(2, 2\pi/3)$. Using Key Idea 9.4.1, we have

$$x = 2\cos(2\pi/3) = -1 \qquad y = 2\sin(2\pi/3) = \sqrt{3}.$$

So the rectangular coordinates are $(-1, \sqrt{3}) \approx (-1, 1.732)$.

(b) The polar point $P(-1, 5\pi/4)$ is converted to rectangular with:

$$x = -1\cos(5\pi/4) = \sqrt{2}/2 \qquad y = -1\sin(5\pi/4) = \sqrt{2}/2.$$

So the rectangular coordinates are $(\sqrt{2}/2, \sqrt{2}/2) \approx (0.707, 0.707)$.

These points are plotted in Figure 9.4.5 (a). The rectangular coordinate system is drawn lightly under the polar coordinate system so that the relationship between the two can be seen.

2. (a) To convert the rectangular point $(1, 2)$ to polar coordinates, we use the Key Idea to form the following two equations:

$$1^2 + 2^2 = r^2 \qquad \tan\theta = \frac{2}{1}.$$

The first equation tells us that $r = \sqrt{5}$. Using the inverse tangent function, we find

$$\tan\theta = 2 \quad \Rightarrow \quad \theta = \tan^{-1} 2 \approx 1.11 \approx 63.43^\circ.$$

Thus polar coordinates of $(1, 2)$ are $P(\sqrt{5}, 1.11)$.

(b) To convert $(-1,1)$ to polar coordinates, we form the equations

$$(-1)^2 + 1^2 = r^2 \qquad \tan\theta = \frac{1}{-1}.$$

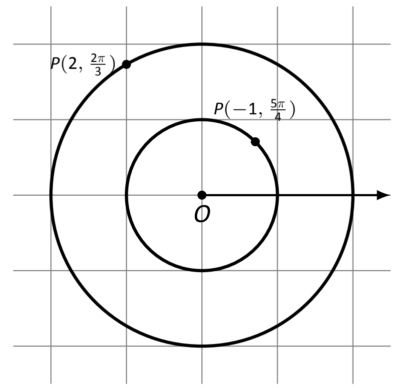
Thus $r = \sqrt{2}$. We need to be careful in computing θ: using the inverse tangent function, we have

$$\tan\theta = -1 \quad \Rightarrow \quad \theta = \tan^{-1}(-1) = -\pi/4 = -45^\circ.$$

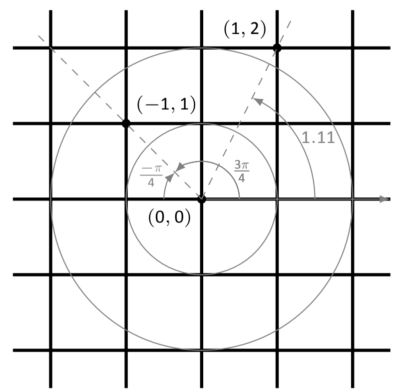
This is not the angle we desire. The range of $\tan^{-1} x$ is $(-\pi/2, \pi/2)$; that is, it returns angles that lie in the 1st and 4th quadrants. To find locations in the 2nd and 3rd quadrants, add π to the result of $\tan^{-1} x$. So $\pi + (-\pi/4)$ puts the angle at $3\pi/4$. Thus the polar point is $P(\sqrt{2}, 3\pi/4)$.

An alternate method is to use the angle θ given by arctangent, but change the sign of r. Thus we could also refer to $(-1,1)$ as $P(-\sqrt{2}, -\pi/4)$.

These points are plotted in Figure 9.4.5 (b). The polar system is drawn lightly under the rectangular grid with rays to demonstrate the angles used.



(a)



(b)

Figure 9.4.5: Plotting rectangular and polar points in Example 9.4.2.

Polar Functions and Polar Graphs

Defining a new coordinate system allows us to create a new kind of function, a **polar function.** Rectangular coordinates lent themselves well to creating functions that related x and y, such as $y = x^2$. Polar coordinates allow us to create functions that relate r and θ. Normally these functions look like $r = f(\theta)$, although we can create functions of the form $\theta = f(r)$. The following examples introduce us to this concept.

Example 9.4.3 Introduction to Graphing Polar Functions

Describe the graphs of the following polar functions.

1. $r = 1.5$
2. $\theta = \pi/4$

Solution

1. The equation $r = 1.5$ describes all points that are 1.5 units from the pole; as the angle is not specified, any θ is allowable. All points 1.5 units from the pole describes a circle of radius 1.5.

   We can consider the rectangular equivalent of this equation; using $r^2 = x^2 + y^2$, we see that $1.5^2 = x^2 + y^2$, which we recognize as the equation of a circle centred at $(0,0)$ with radius 1.5. This is sketched in Figure 9.4.6.

2. The equation $\theta = \pi/4$ describes all points such that the line through them and the pole make an angle of $\pi/4$ with the initial ray. As the radius r is not specified, it can be any value (even negative). Thus $\theta = \pi/4$ describes the line through the pole that makes an angle of $\pi/4 = 45^\circ$ with the initial ray.

   We can again consider the rectangular equivalent of this equation. Combine $\tan\theta = y/x$ and $\theta = \pi/4$:

$$\tan\pi/4 = y/x \quad \Rightarrow x\tan\pi/4 = y \quad \Rightarrow y = x.$$

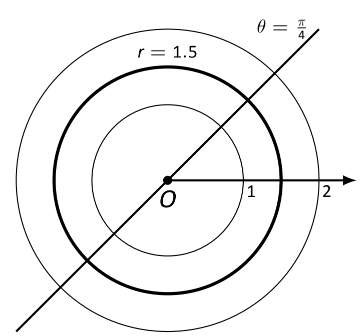


Figure 9.4.6: Plotting standard polar plots.

The basic rectangular equations of the form $x = h$ and $y = k$ create vertical and horizontal lines, respectively; the basic polar equations $r = h$ and $\theta = \alpha$ create circles and lines through the pole, respectively. With this as a foundation, we can create more complicated polar functions of the form $r = f(\theta)$. The input is an angle; the output is a length, how far in the direction of the angle to go out.

We sketch these functions much like we sketch rectangular and parametric functions: we plot lots of points and "connect the dots" with curves. We demonstrate this in the following example.

θ	$r = 1 + \cos\theta$
0	2
$\pi/6$	1.86603
$\pi/2$	1
$4\pi/3$	0.5
$7\pi/4$	1.70711

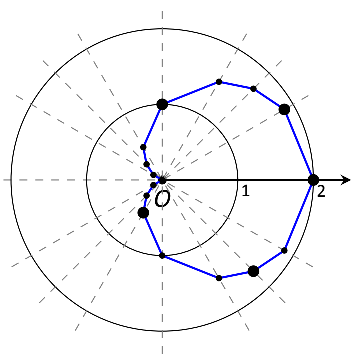


Figure 9.4.8: Graphing a polar function in Example 9.4.4 by plotting points.

Example 9.4.4 Sketching Polar Functions
Sketch the polar function $r = 1 + \cos\theta$ on $[0, 2\pi]$ by plotting points.

Solution A common question when sketching curves by plotting points is "Which points should I plot?" With rectangular equations, we often choose "easy" values – integers, then add more if needed. When plotting polar equations, start with the "common" angles – multiples of $\pi/6$ and $\pi/4$. Figure 9.4.8 gives a table of just a few values of θ in $[0, \pi]$.

Consider the point $P(0, 2)$ determined by the first line of the table. The angle is 0 radians – we do not rotate from the initial ray – then we go out 2 units from the pole. When $\theta = \pi/6$, $r = 1.866$ (actually, it is $1 + \sqrt{3}/2$); so rotate by $\pi/6$ radians and go out 1.866 units.

The graph shown uses more points, connected with straight lines. (The points on the graph that correspond to points in the table are signified with larger dots.) Such a sketch is likely good enough to give one an idea of what the graph looks like.

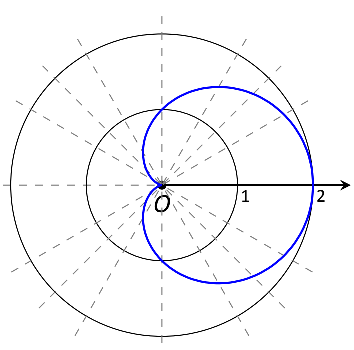


Figure 9.4.9: Using technology to graph a polar function.

Technology Note: Plotting functions in this way can be tedious, just as it was with rectangular functions. To obtain very accurate graphs, technology is a great aid. Most graphing calculators can plot polar functions; in the menu, set the plotting mode to something like `polar` or `POL`, depending on one's calculator. As with plotting parametric functions, the viewing "window" no longer determines the x-values that are plotted, so additional information needs to be provided. Often with the "window" settings are the settings for the beginning and ending θ values (often called $\theta_{\texttt{min}}$ and $\theta_{\texttt{max}}$) as well as the $\theta_{\texttt{step}}$ – that is, how far apart the θ values are spaced. The smaller the $\theta_{\texttt{step}}$ value, the more accurate the graph (which also increases plotting time). Using technology, we graphed the polar function $r = 1 + \cos\theta$ from Example 9.4.4 in Figure 9.4.9.

Example 9.4.5 Sketching Polar Functions
Sketch the polar function $r = \cos(2\theta)$ on $[0, 2\pi]$ by plotting points.

Solution We start by making a table of $\cos(2\theta)$ evaluated at common angles θ, as shown in Figure 9.4.7. These points are then plotted in Figure 9.4.10 (a). This particular graph "moves" around quite a bit and one can easily forget which points should be connected to each other. To help us with this, we numbered each point in the table and on the graph.

Pt.	θ	$\cos(2\theta)$	Pt.	θ	$\cos(2\theta)$
1	0	1.	10	$7\pi/6$	0.5
2	$\pi/6$	0.5	11	$5\pi/4$	0.
3	$\pi/4$	0.	12	$4\pi/3$	-0.5
4	$\pi/3$	-0.5	13	$3\pi/2$	$-1.$
5	$\pi/2$	$-1.$	14	$5\pi/3$	-0.5
6	$2\pi/3$	-0.5	15	$7\pi/4$	0.
7	$3\pi/4$	0.	16	$11\pi/6$	0.5
8	$5\pi/6$	0.5	17	2π	1.
9	π	1.			

Figure 9.4.9: Tables of points for plotting a polar curve.

Using more points (and the aid of technology) a smoother plot can be made as shown in Figure 9.4.10 (b). This plot is an example of a *rose curve*.

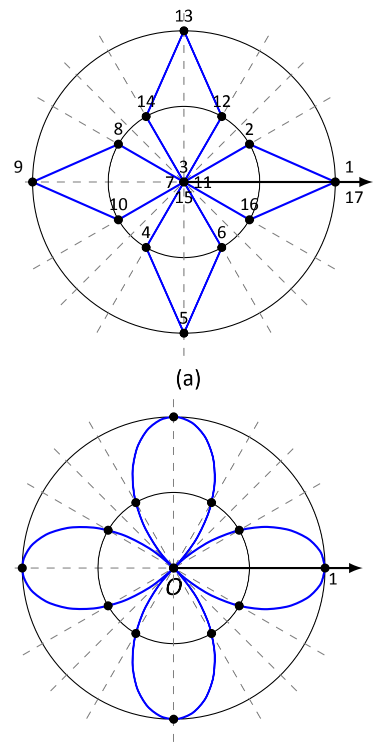


Figure 9.4.10: Polar plots from Example 9.4.5.

It is sometimes desirable to refer to a graph via a polar equation, and other times by a rectangular equation. Therefore it is necessary to be able to convert between polar and rectangular functions, which we practice in the following example. We will make frequent use of the identities found in Key Idea 9.4.1.

Example 9.4.6 Converting between rectangular and polar equations.

Convert from rectangular to polar.

1. $y = x^2$
2. $xy = 1$

Convert from polar to rectangular.

3. $r = \dfrac{2}{\sin\theta - \cos\theta}$
4. $r = 2\cos\theta$

SOLUTION

1. Replace y with $r\sin\theta$ and replace x with $r\cos\theta$, giving:

$$\begin{aligned} y &= x^2 \\ r\sin\theta &= r^2\cos^2\theta \\ \frac{\sin\theta}{\cos^2\theta} &= r \end{aligned}$$

We have found that $r = \sin\theta/\cos^2\theta = \tan\theta\sec\theta$. The domain of this polar function is $(-\pi/2, \pi/2)$; plot a few points to see how the familiar parabola is traced out by the polar equation.

2. We again replace x and y using the standard identities and work to solve for r:

$$\begin{aligned} xy &= 1 \\ r\cos\theta \cdot r\sin\theta &= 1 \\ r^2 &= \frac{1}{\cos\theta\sin\theta} \\ r &= \frac{1}{\sqrt{\cos\theta\sin\theta}} \end{aligned}$$

This function is valid only when the product of $\cos\theta\sin\theta$ is positive. This occurs in the first and third quadrants, meaning the domain of this polar function is $(0, \pi/2) \cup (\pi, 3\pi/2)$.

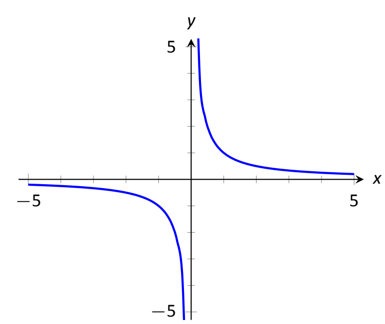


Figure 9.4.11: Graphing $xy = 1$ from Example 9.4.6.

We can rewrite the original rectangular equation $xy = 1$ as $y = 1/x$. This is graphed in Figure 9.4.11; note how it only exists in the first and third quadrants.

3. There is no set way to convert from polar to rectangular; in general, we look to form the products $r\cos\theta$ and $r\sin\theta$, and then replace these with x and y, respectively. We start in this problem by multiplying both sides by $\sin\theta - \cos\theta$:

$$\begin{aligned} r &= \frac{2}{\sin\theta - \cos\theta} \\ r(\sin\theta - \cos\theta) &= 2 \\ r\sin\theta - r\cos\theta &= 2. \qquad \text{Now replace with } y \text{ and } x\text{:} \\ y - x &= 2 \\ y &= x + 2. \end{aligned}$$

The original polar equation, $r = 2/(\sin\theta - \cos\theta)$ does not easily reveal that its graph is simply a line. However, our conversion shows that it is. The upcoming gallery of polar curves gives the general equations of lines in polar form.

4. By multiplying both sides by r, we obtain both an r^2 term and an $r\cos\theta$ term, which we replace with $x^2 + y^2$ and x, respectively.

$$\begin{aligned} r &= 2\cos\theta \\ r^2 &= 2r\cos\theta \\ x^2 + y^2 &= 2x. \end{aligned}$$

We recognize this as a circle; by completing the square we can find its radius and center.

$$\begin{aligned} x^2 - 2x + y^2 &= 0 \\ (x-1)^2 + y^2 &= 1. \end{aligned}$$

The circle is centered at $(1, 0)$ and has radius 1. The upcoming gallery of polar curves gives the equations of *some* circles in polar form; circles with arbitrary centers have a complicated polar equation that we do not consider here.

Some curves have very simple polar equations but rather complicated rectangular ones. For instance, the equation $r = 1 + \cos\theta$ describes a *cardioid* (a shape important to the sensitivity of microphones, among other things; one is graphed in the gallery in the Limaçon section). It's rectangular form is not nearly as simple; it is the implicit equation $x^4 + y^4 + 2x^2y^2 - 2xy^2 - 2x^3 - y^2 = 0$. The conversion is not "hard," but takes several steps, and is left as a problem in the Exercise section.

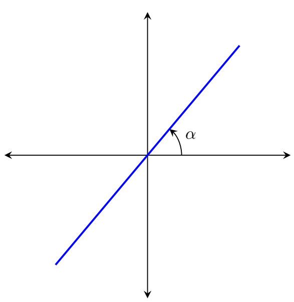
Gallery of Polar Curves

There are a number of basic and "classic" polar curves, famous for their beauty and/or applicability to the sciences. This section ends with a small gallery of some of these graphs. We encourage the reader to understand how these graphs are formed, and to investigate with technology other types of polar functions.

Lines

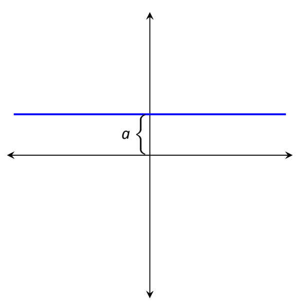
Through the origin:

$\theta = \alpha$



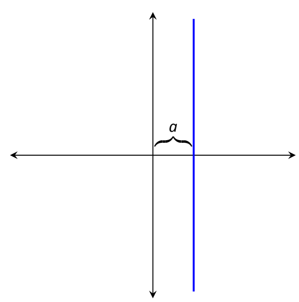
Horizontal line:

$r = a\csc\theta$



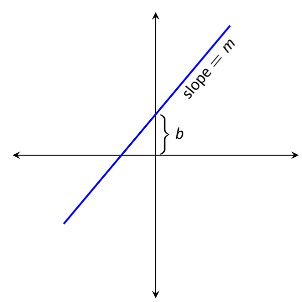
Vertical line:

$r = a\sec\theta$



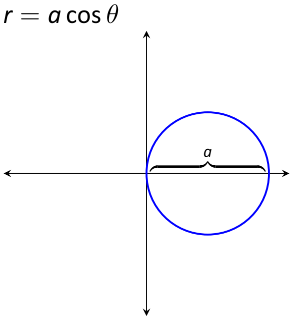
Not through origin:

$$r = \frac{b}{\sin\theta - m\cos\theta}$$

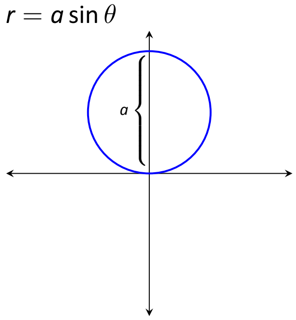


Circles

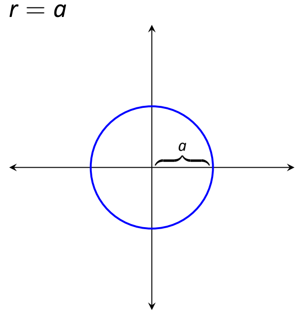
Centered on *x*-axis:
$r = a\cos\theta$



Centered on *y*-axis:
$r = a\sin\theta$

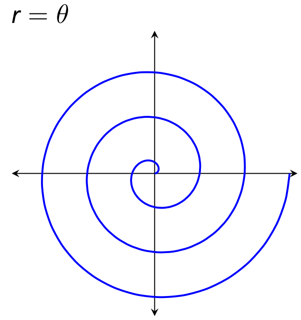


Centered on origin:
$r = a$



Spiral

Archimedean spiral
$r = \theta$

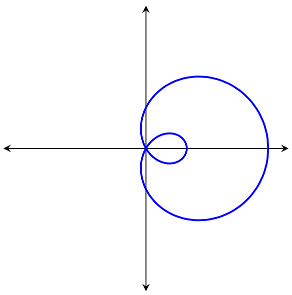


Limaçons

Symmetric about x-axis: $r = a \pm b\cos\theta$; Symmetric about y-axis: $r = a \pm b\sin\theta$; $a, b > 0$

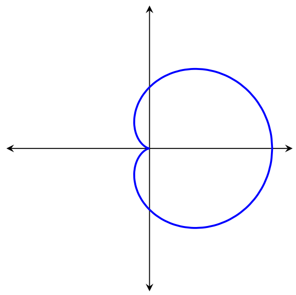
With inner loop:

$\frac{a}{b} < 1$



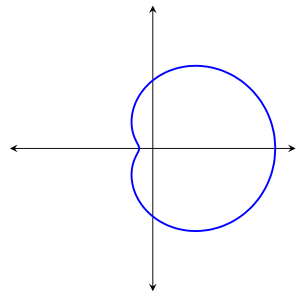
Cardioid:

$\frac{a}{b} = 1$



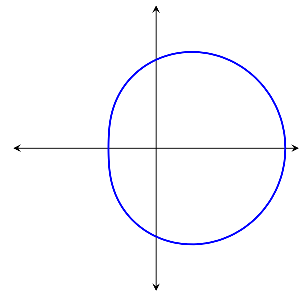
Dimpled:

$1 < \frac{a}{b} < 2$



Convex:

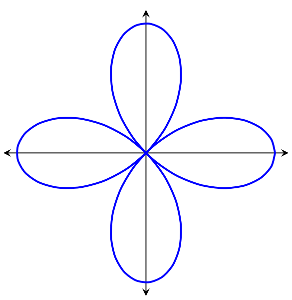
$\frac{a}{b} > 2$



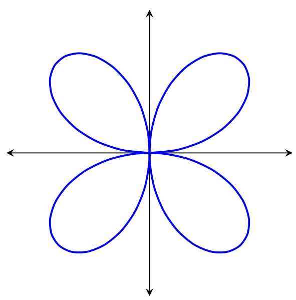
Rose Curves

Symmetric about x-axis: $r = a\cos(n\theta)$; Symmetric about y-axis: $r = a\sin(n\theta)$
Curve contains $2n$ petals when n is even and n petals when n is odd.

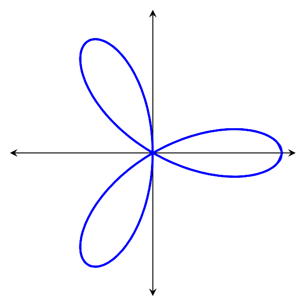
$r = a\cos(2\theta)$



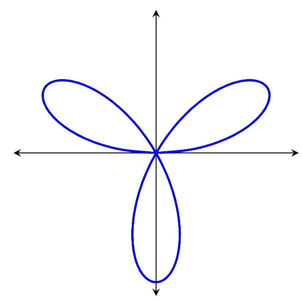
$r = a\sin(2\theta)$



$r = a\cos(3\theta)$



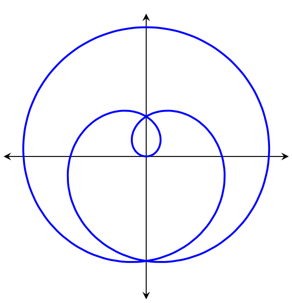
$r = a\sin(3\theta)$



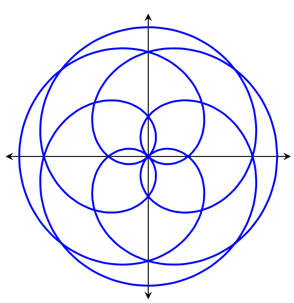
Special Curves

Rose curves

$r = a\sin(\theta/5)$

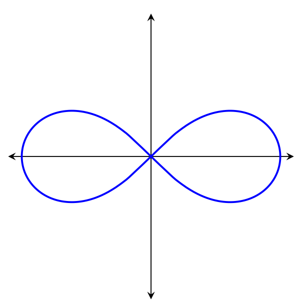


$r = a\sin(2\theta/5)$



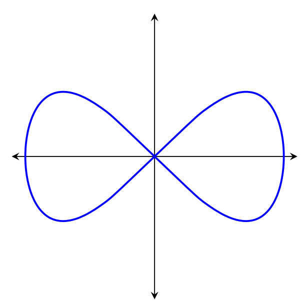
Lemniscate:

$r^2 = a^2\cos(2\theta)$



Eight Curve:

$r^2 = a^2\sec^4\theta\cos(2\theta)$



Earlier we discussed how each point in the plane does not have a unique representation in polar form. This can be a "good" thing, as it allows for the beautiful and interesting curves seen in the preceding gallery. However, it can also be a "bad" thing, as it can be difficult to determine where two curves intersect.

Example 9.4.7 Finding points of intersection with polar curves

Determine where the graphs of the polar equations $r = 1+3\cos\theta$ and $r = \cos\theta$ intersect.

Solution As technology is generally readily available, it is usually a good idea to start with a graph. We have graphed the two functions in Figure 9.4.12(a); to better discern the intersection points, part (b) of the figure zooms in around the origin. We start by setting the two functions equal to each other and solving for θ:

$$\begin{aligned} 1+3\cos\theta &= \cos\theta \\ 2\cos\theta &= -1 \\ \cos\theta &= -\frac{1}{2} \\ \theta &= \frac{2\pi}{3}, \frac{4\pi}{3}. \end{aligned}$$

(There are, of course, infinite solutions to the equation $\cos\theta = -1/2$; as the limaçon is traced out once on $[0, 2\pi]$, we restrict our solutions to this interval.)

We need to analyze this solution. When $\theta = 2\pi/3$ we obtain the point of intersection that lies in the 4$^{\text{th}}$ quadrant. When $\theta = 4\pi/3$, we get the point of intersection that lies in the 2$^{\text{nd}}$ quadrant. There is more to say about this second intersection point, however. The circle defined by $r = \cos\theta$ is traced out once on $[0, \pi]$, meaning that this point of intersection occurs while tracing out the circle a second time. It seems strange to pass by the point once and then recognize it as a point of intersection only when arriving there a "second time." The first time the circle arrives at this point is when $\theta = \pi/3$. It is key to understand that these two points are the same: $(\cos\pi/3, \pi/3)$ and $(\cos 4\pi/3, 4\pi/3)$.

To summarize what we have done so far, we have found two points of intersection: when $\theta = 2\pi/3$ and when $\theta = 4\pi/3$. When referencing the circle $r = \cos\theta$, the latter point is better referenced as when $\theta = \pi/3$.

There is yet another point of intersection: the pole (or, the origin). We did not recognize this intersection point using our work above as each graph arrives at the pole at a different θ value.

A graph intersects the pole when $r = 0$. Considering the circle $r = \cos\theta$, $r = 0$ when $\theta = \pi/2$ (and odd multiples thereof, as the circle is repeatedly traced). The limaçon intersects the pole when $1+3\cos\theta = 0$; this occurs when $\cos\theta = -1/3$, or for $\theta = \cos^{-1}(-1/3)$. This is a nonstandard angle, approximately $\theta = 1.9106 = 109.47^\circ$. The limaçon intersects the pole twice in $[0, 2\pi]$; the other angle at which the limaçon is at the pole is the reflection of the first angle across the x-axis. That is, $\theta = 4.3726 = 250.53^\circ$.

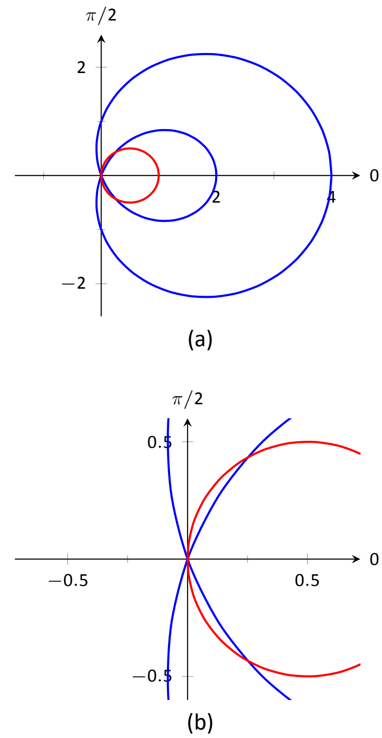


Figure 9.4.12: Graphs to help determine the points of intersection of the polar functions given in Example 9.4.7.

If all one is concerned with is the (x, y) coordinates at which the graphs intersect, much of the above work is extraneous. We know they intersect at $(0, 0)$; we might not care at what θ value. Likewise, using $\theta = 2\pi/3$ and $\theta = 4\pi/3$ can give us the needed rectangular coordinates. However, in the next section we apply calculus concepts to polar functions. When computing the area of a region bounded by polar curves, understanding the nuances of the points of intersection becomes important.

Exercises 9.4

Terms and Concepts

1. In your own words, describe how to plot the polar point $P(r,\theta)$.

2. T/F: When plotting a point with polar coordinate $P(r,\theta)$, r must be positive.

3. T/F: Every point in the Cartesian plane can be represented by a polar coordinate.

4. T/F: Every point in the Cartesian plane can be represented uniquely by a polar coordinate.

Problems

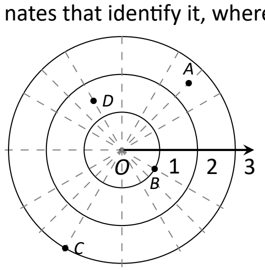
5. Plot the points with the given polar coordinates.

   (a) $A = P(2, 0)$ (b) $B = P(1, \pi)$ (c) $C = P(-2, \pi/2)$ (d) $D = P(1, \pi/4)$

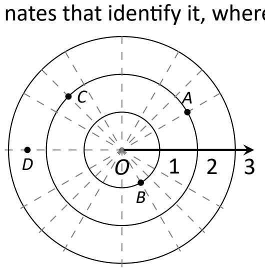
6. Plot the points with the given polar coordinates.

   (a) $A = P(2, 3\pi)$ (b) $B = P(1, -\pi)$ (c) $C = P(1, 2)$ (d) $D = P(1/2, 5\pi/6)$

7. For each of the given points give two sets of polar coordinates that identify it, where $0 \leq \theta \leq 2\pi$.



8. For each of the given points give two sets of polar coordinates that identify it, where $-\pi \leq \theta \leq \pi$.



9. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

   (a) $A = P(2, \pi/4)$ (b) $B = P(2, -\pi/4)$ (c) $C = (2, -1)$ (d) $D = (-2, 1)$

10. Convert each of the following polar coordinates to rectangular, and each of the following rectangular coordinates to polar.

   (a) $A = P(3, \pi)$ (b) $B = P(1, 2\pi/3)$ (c) $C = (0, 4)$ (d) $D = (1, -\sqrt{3})$

In Exercises 11 – 30, graph the polar function on the given interval.

11. $r = 2, \quad 0 \leq \theta \leq \pi/2$

12. $\theta = \pi/6, \quad -1 \leq r \leq 2$

13. $r = 1 - \cos\theta, \quad [0, 2\pi]$

14. $r = 2 + \sin\theta, \quad [0, 2\pi]$

15. $r = 2 - \sin\theta, \quad [0, 2\pi]$

16. $r = 1 - 2\sin\theta, \quad [0, 2\pi]$

17. $r = 1 + 2\sin\theta, \quad [0, 2\pi]$

18. $r = \cos(2\theta), \quad [0, 2\pi]$

19. $r = \sin(3\theta), \quad [0, \pi]$

20. $r = \cos(\theta/3), \quad [0, 3\pi]$

21. $r = \cos(2\theta/3), \quad [0, 6\pi]$

22. $r = \theta/2, \quad [0, 4\pi]$

23. $r = 3\sin(\theta), \quad [0, \pi]$

24. $r = 2\cos(\theta), \quad [0, \pi/2]$

25. $r = \cos\theta\sin\theta, \quad [0, 2\pi]$

26. $r = \theta^2 - (\pi/2)^2, \quad [-\pi, \pi]$

27. $r = \dfrac{3}{5\sin\theta - \cos\theta}, \quad [0, 2\pi]$

28. $r = \dfrac{-2}{3\cos\theta - 2\sin\theta}, \quad [0, 2\pi]$

29. $r = 3\sec\theta, \quad (-\pi/2, \pi/2)$

30. $r = 3\csc\theta, \quad (0, \pi)$

In Exercises 31 – 40, convert the polar equation to a rectangular equation.

31. $r = 6\cos\theta$

32. $r = -4\sin\theta$

33. $r = \cos\theta + \sin\theta$

34. $r = \dfrac{7}{5\sin\theta - 2\cos\theta}$

35. $r = \dfrac{3}{\cos\theta}$

36. $r = \dfrac{4}{\sin\theta}$

37. $r = \tan\theta$

38. $r = \cot\theta$

39. $r = 2$

40. $\theta = \pi/6$

In Exercises 41 – 48, convert the rectangular equation to a polar equation.

41. $y = x$

42. $y = 4x + 7$

43. $x = 5$

44. $y = 5$

45. $x = y^2$

46. $x^2y = 1$

47. $x^2 + y^2 = 7$

48. $(x+1)^2 + y^2 = 1$

In Exercises 49 – 56, find the points of intersection of the polar graphs.

49. $r = \sin(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

50. $r = \cos(2\theta)$ and $r = \cos\theta$ on $[0, \pi]$

51. $r = 2\cos\theta$ and $r = 2\sin\theta$ on $[0, \pi]$

52. $r = \sin\theta$ and $r = \sqrt{3} + 3\sin\theta$ on $[0, 2\pi]$

53. $r = \sin(3\theta)$ and $r = \cos(3\theta)$ on $[0, \pi]$

54. $r = 3\cos\theta$ and $r = 1 + \cos\theta$ on $[-\pi, \pi]$

55. $r = 1$ and $r = 2\sin(2\theta)$ on $[0, 2\pi]$

56. $r = 1 - \cos\theta$ and $r = 1 + \sin\theta$ on $[0, 2\pi]$

57. Pick a integer value for n, where $n \neq 2, 3$, and use technology to plot $r = \sin\left(\dfrac{m}{n}\theta\right)$ for three different integer values of m. Sketch these and determine a minimal interval on which the entire graph is shown.

58. Create your own polar function, $r = f(\theta)$ and sketch it. Describe why the graph looks as it does.

9.5 Calculus and Polar Functions

The previous section defined polar coordinates, leading to polar functions. We investigated plotting these functions and solving a fundamental question about their graphs, namely, where do two polar graphs intersect?

We now turn our attention to answering other questions, whose solutions require the use of calculus. A basis for much of what is done in this section is the ability to turn a polar function $r = f(\theta)$ into a set of parametric equations. Using the identities $x = r\cos\theta$ and $y = r\sin\theta$, we can create the parametric equations $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ and apply the concepts of Section 9.3.

Polar Functions and $\dfrac{dy}{dx}$

We are interested in the lines tangent to a given graph, regardless of whether that graph is produced by rectangular, parametric, or polar equations. In each of these contexts, the slope of the tangent line is $\frac{dy}{dx}$. Given $r = f(\theta)$, we are generally *not* concerned with $r' = f'(\theta)$; that describes how fast r changes with respect to θ. Instead, we will use $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ to compute $\frac{dy}{dx}$.

Using Key Idea 9.3.1 we have

$$\frac{dy}{dx} = \frac{dy}{d\theta}\bigg/\frac{dx}{d\theta}.$$

Each of the two derivatives on the right hand side of the equality requires the use of the Product Rule. We state the important result as a Key Idea.

Key Idea 9.5.1 Finding $\frac{dy}{dx}$ with Polar Functions

Let $r = f(\theta)$ be a polar function. With $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$,

$$\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}.$$

Example 9.5.1 Finding $\frac{dy}{dx}$ with polar functions.

Consider the limaçon $r = 1 + 2\sin\theta$ on $[0, 2\pi]$.

1. Find the equations of the tangent and normal lines to the graph at $\theta = \pi/4$.

2. Find where the graph has vertical and horizontal tangent lines.

Solution

1. We start by computing $\frac{dy}{dx}$. With $f'(\theta) = 2\cos\theta$, we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{2\cos\theta\sin\theta + \cos\theta(1 + 2\sin\theta)}{2\cos^2\theta - \sin\theta(1 + 2\sin\theta)}\\ &= \frac{\cos\theta(4\sin\theta + 1)}{2(\cos^2\theta - \sin^2\theta) - \sin\theta}.\end{aligned}$$

When $\theta = \pi/4$, $\frac{dy}{dx} = -2\sqrt{2} - 1$ (this requires a bit of simplification). In rectangular coordinates, the point on the graph at $\theta = \pi/4$ is $(1 +$

$\sqrt{2}/2, 1+\sqrt{2}/2)$. Thus the rectangular equation of the line tangent to the limaçon at $\theta = \pi/4$ is

$$y = (-2\sqrt{2}-1)\big(x-(1+\sqrt{2}/2)\big)+1+\sqrt{2}/2 \approx -3.83x+8.24.$$

The limaçon and the tangent line are graphed in Figure 9.5.1.

The normal line has the opposite–reciprocal slope as the tangent line, so its equation is

$$y \approx \frac{1}{3.83}x + 1.26.$$

2. To find the horizontal lines of tangency, we find where $\frac{dy}{dx} = 0$; thus we find where the numerator of our equation for $\frac{dy}{dx}$ is 0.

$$\cos\theta(4\sin\theta+1) = 0 \quad\Rightarrow\quad \cos\theta = 0 \quad\text{or}\quad 4\sin\theta+1=0.$$

On $[0, 2\pi]$, $\cos\theta = 0$ when $\theta = \pi/2,\ 3\pi/2$.

Setting $4\sin\theta + 1 = 0$ gives $\theta = \sin^{-1}(-1/4) \approx -0.2527 = -14.48°$. We want the results in $[0, 2\pi]$; we also recognize there are two solutions, one in the 3$^{\text{rd}}$ quadrant and one in the 4$^{\text{th}}$. Using reference angles, we have our two solutions as $\theta = 3.39$ and 6.03 radians. The four points we obtained where the limaçon has a horizontal tangent line are given in Figure 9.5.1 with black–filled dots.

To find the vertical lines of tangency, we set the denominator of $\frac{dy}{dx} = 0$.

$$2(\cos^2\theta - \sin^2\theta) - \sin\theta = 0.$$

Convert the $\cos^2\theta$ term to $1-\sin^2\theta$:

$$\begin{aligned} 2(1-\sin^2\theta-\sin^2\theta)-\sin\theta &= 0\\ 4\sin^2\theta+\sin\theta-2 &= 0. \end{aligned}$$

Recognize this as a quadratic in the variable $\sin\theta$. Using the quadratic formula, we have

$$\sin\theta = \frac{-1\pm\sqrt{33}}{8}.$$

We solve $\sin\theta = \frac{-1+\sqrt{33}}{8}$ and $\sin\theta = \frac{-1-\sqrt{33}}{8}$:

$$\begin{aligned} \sin\theta &= \frac{-1+\sqrt{33}}{8} & \qquad \sin\theta &= \frac{-1-\sqrt{33}}{8}\\ \theta &= \sin^{-1}\left(\frac{-1+\sqrt{33}}{8}\right) & \theta &= \sin^{-1}\left(\frac{-1-\sqrt{33}}{8}\right)\\ \theta &= 0.6349 & \theta &= -1.0030 \end{aligned}$$

In each of the solutions above, we only get one of the possible two solutions as $\sin^{-1}x$ only returns solutions in $[-\pi/2, \pi/2]$, the 4$^{\text{th}}$ and 1$^{\text{st}}$ quadrants. Again using reference angles, we have:

$$\sin\theta = \frac{-1+\sqrt{33}}{8} \quad\Rightarrow\quad \theta = 0.6349,\ 2.5067 \text{ radians}$$

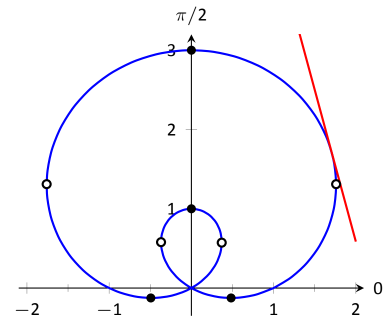


Figure 9.5.1: The limaçon in Example 9.5.1 with its tangent line at $\theta = \pi/4$ and points of vertical and horizontal tangency.

and

$$\sin\theta = \frac{-1-\sqrt{33}}{8} \quad \Rightarrow \quad \theta = 4.1446,\ 5.2802 \text{ radians.}$$

These points are also shown in Figure 9.5.1 with white–filled dots.

When the graph of the polar function $r = f(\theta)$ intersects the pole, it means that $f(\alpha) = 0$ for some angle α. Thus the formula for $\frac{dy}{dx}$ in such instances is very simple, reducing simply to

$$\frac{dy}{dx} = \tan\alpha.$$

This equation makes an interesting point. It tells us the slope of the tangent line at the pole is $\tan\alpha$; some of our previous work (see, for instance, Example 9.4.3) shows us that the line through the pole with slope $\tan\alpha$ has polar equation $\theta = \alpha$. Thus when a polar graph touches the pole at $\theta = \alpha$, the equation of the tangent line at the pole is $\theta = \alpha$.

Example 9.5.2 Finding tangent lines at the pole.

Let $r = 1 + 2\sin\theta$, a limaçon. Find the equations of the lines tangent to the graph at the pole.

SOLUTION We need to know when $r = 0$.

$$\begin{aligned} 1 + 2\sin\theta &= 0 \\ \sin\theta &= -1/2 \\ \theta &= \frac{7\pi}{6},\ \frac{11\pi}{6}. \end{aligned}$$

Thus the equations of the tangent lines, in polar, are $\theta = 7\pi/6$ and $\theta = 11\pi/6$. In rectangular form, the tangent lines are $y = \tan(7\pi/6)x$ and $y = \tan(11\pi/6)x$. The full limaçon can be seen in Figure 9.5.1; we zoom in on the tangent lines in Figure 9.5.2.

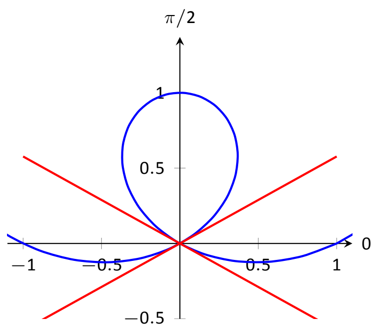


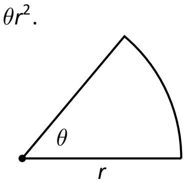
Figure 9.5.2: Graphing the tangent lines at the pole in Example 9.5.2.

Area

When using rectangular coordinates, the equations $x = h$ and $y = k$ defined vertical and horizontal lines, respectively, and combinations of these lines create rectangles (hence the name "rectangular coordinates"). It is then somewhat natural to use rectangles to approximate area as we did when learning about the definite integral.

When using polar coordinates, the equations $\theta = \alpha$ and $r = c$ form lines through the origin and circles centred at the origin, respectively, and combinations of these curves form sectors of circles. It is then somewhat natural to calculate the area of regions defined by polar functions by first approximating with sectors of circles.

Note: Recall that the area of a sector of a circle with radius r subtended by an angle θ is $A = \frac{1}{2}\theta r^2$.



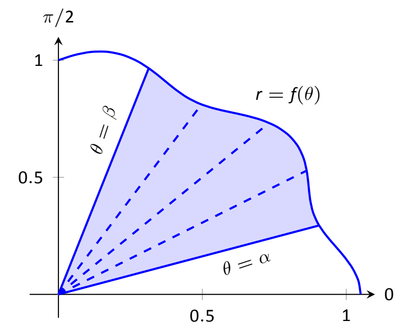
Consider Figure 9.5.3 (a) where a region defined by $r = f(\theta)$ on $[\alpha, \beta]$ is given. (Note how the "sides" of the region are the lines $\theta = \alpha$ and $\theta = \beta$, whereas in rectangular coordinates the "sides" of regions were often the vertical lines $x = a$ and $x = b$.)

Partition the interval $[\alpha, \beta]$ into n equally spaced subintervals as $\alpha = \theta_1 < \theta_2 < \cdots < \theta_{n+1} = \beta$. The length of each subinterval is $\Delta\theta = (\beta - \alpha)/n$, representing a small change in angle. The area of the region defined by the i^{th} subinterval $[\theta_i, \theta_{i+1}]$ can be approximated with a sector of a circle with radius

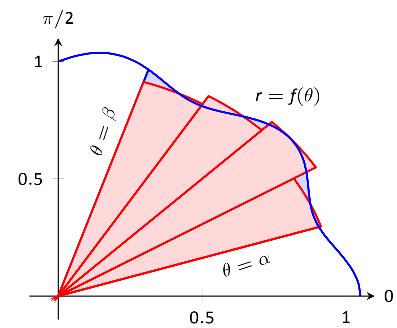
$f(c_i)$, for some c_i in $[\theta_i, \theta_{i+1}]$. The area of this sector is $\frac{1}{2}f(c_i)^2\Delta\theta$. This is shown in part (b) of the figure, where $[\alpha, \beta]$ has been divided into 4 subintervals. We approximate the area of the whole region by summing the areas of all sectors:

$$\text{Area} \approx \sum_{i=1}^{n} \frac{1}{2} f(c_i)^2 \Delta\theta.$$

This is a Riemann sum. By taking the limit of the sum as $n \to \infty$, we find the exact area of the region in the form of a definite integral.



(a)



(b)

Figure 9.5.3: Computing the area of a polar region.

Theorem 9.5.1 Area of a Polar Region

Let f be continuous and non-negative on $[\alpha, \beta]$, where $0 \leq \beta - \alpha \leq 2\pi$. The area A of the region bounded by the curve $r = f(\theta)$ and the lines $\theta = \alpha$ and $\theta = \beta$ is

$$A = \frac{1}{2}\int_\alpha^\beta f(\theta)^2\, d\theta = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

The theorem states that $0 \leq \beta - \alpha \leq 2\pi$. This ensures that region does not overlap itself, which would give a result that does not correspond directly to the area.

Example 9.5.3 Area of a polar region

Find the area of the circle defined by $r = \cos\theta$. (Recall this circle has radius $1/2$.)

Solution This is a direct application of Theorem 9.5.1. The circle is traced out on $[0, \pi]$, leading to the integral

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_0^\pi \cos^2\theta\, d\theta\\ &= \frac{1}{2}\int_0^\pi \frac{1+\cos(2\theta)}{2}\, d\theta\\ &= \frac{1}{4}\left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_0^\pi\\ &= \frac{1}{4}\pi.\end{aligned}$$

Of course, we already knew the area of a circle with radius $1/2$. We did this example to demonstrate that the area formula is correct.

Note: Example 9.5.3 requires the use of the integral $\int \cos^2\theta\, d\theta$. This is handled well by using the power reducing formula as found at the end of this text. Due to the nature of the area formula, integrating $\cos^2\theta$ and $\sin^2\theta$ is required often. We offer here these indefinite integrals as a time–saving measure.

$$\int \cos^2\theta\, d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$
$$\int \sin^2\theta\, d\theta = \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + C$$

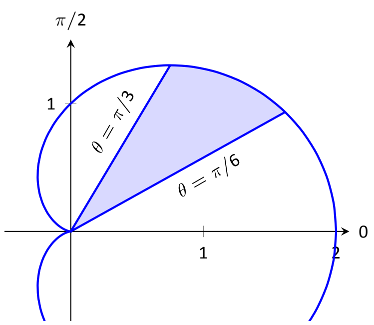


Figure 9.5.4: Finding the area of the shaded region of a cardiod in Example 9.5.4.

Example 9.5.4 Area of a polar region

Find the area of the cardioid $r = 1+\cos\theta$ bound between $\theta = \pi/6$ and $\theta = \pi/3$, as shown in Figure 9.5.4.

SOLUTION This is again a direct application of Theorem 9.5.1.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{\pi/6}^{\pi/3}(1+\cos\theta)^2\,d\theta\\ &= \frac{1}{2}\int_{\pi/6}^{\pi/3}(1+2\cos\theta+\cos^2\theta)\,d\theta\\ &= \frac{1}{2}\left(\theta+2\sin\theta+\frac{1}{2}\theta+\frac{1}{4}\sin(2\theta)\right)\Bigg|_{\pi/6}^{\pi/3}\\ &= \frac{1}{8}\left(\pi+4\sqrt{3}-4\right)\approx 0.7587.\end{aligned}$$

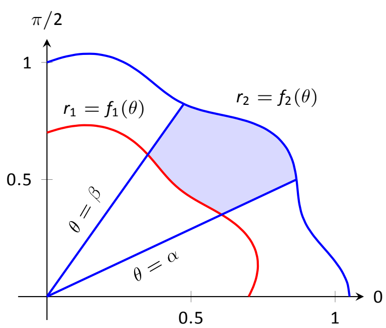


Figure 9.5.5: Illustrating area bound between two polar curves.

Area Between Curves

Our study of area in the context of rectangular functions led naturally to finding area bounded between curves. We consider the same in the context of polar functions.

Consider the shaded region shown in Figure 9.5.5. We can find the area of this region by computing the area bounded by $r_2 = f_2(\theta)$ and subtracting the area bounded by $r_1 = f_1(\theta)$ on $[\alpha, \beta]$. Thus

$$\text{Area} = \frac{1}{2}\int_\alpha^\beta r_2^2\,d\theta - \frac{1}{2}\int_\alpha^\beta r_1^2\,d\theta = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right)\,d\theta.$$

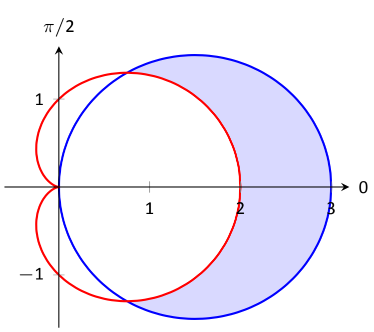


Figure 9.5.6: Finding the area between polar curves in Example 9.5.5.

Key Idea 9.5.2 Area Between Polar Curves

The area A of the region bounded by $r_1 = f_1(\theta)$ and $r_2 = f_2(\theta)$, $\theta = \alpha$ and $\theta = \beta$, where $f_1(\theta) \le f_2(\theta)$ on $[\alpha, \beta]$, is

$$A = \frac{1}{2}\int_\alpha^\beta \left(r_2^2 - r_1^2\right)\,d\theta.$$

Example 9.5.5 Area between polar curves

Find the area bounded between the curves $r = 1+\cos\theta$ and $r = 3\cos\theta$, as shown in Figure 9.5.6.

SOLUTION We need to find the points of intersection between these two functions. Setting them equal to each other, we find:

$$\begin{aligned}1+\cos\theta &= 3\cos\theta\\ \cos\theta &= 1/2\\ \theta &= \pm\pi/3\end{aligned}$$

Thus we integrate $\frac{1}{2}\left((3\cos\theta)^2-(1+\cos\theta)^2\right)$ on $[-\pi/3,\pi/3]$.

$$\begin{aligned}\text{Area} &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left((3\cos\theta)^2-(1+\cos\theta)^2\right)\,d\theta\\ &= \frac{1}{2}\int_{-\pi/3}^{\pi/3}\left(8\cos^2\theta-2\cos\theta-1\right)\,d\theta\\ &= \frac{1}{2}\big(2\sin(2\theta)-2\sin\theta+3\theta\big)\Big|_{-\pi/3}^{\pi/3}\\ &= \pi.\end{aligned}$$

Amazingly enough, the area between these curves has a "nice" value.

Example 9.5.6 Area defined by polar curves

Find the area bounded between the polar curves $r=1$ and $r=2\cos(2\theta)$, as shown in Figure 9.5.7 (a).

SOLUTION We need to find the point of intersection between the two curves. Setting the two functions equal to each other, we have

$$2\cos(2\theta)=1 \quad\Rightarrow\quad \cos(2\theta)=\frac{1}{2} \quad\Rightarrow\quad 2\theta=\pi/3 \quad\Rightarrow\quad \theta=\pi/6.$$

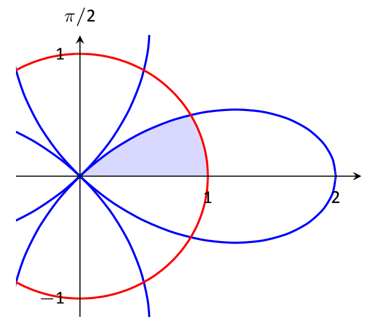
In part (b) of the figure, we zoom in on the region and note that it is not really bounded *between* two polar curves, but rather *by* two polar curves, along with $\theta=0$. The dashed line breaks the region into its component parts. Below the dashed line, the region is defined by $r=1$, $\theta=0$ and $\theta=\pi/6$. (Note: the dashed line lies on the line $\theta=\pi/6$.) Above the dashed line the region is bounded by $r=2\cos(2\theta)$ and $\theta=\pi/6$. Since we have two separate regions, we find the area using two separate integrals.

Call the area below the dashed line A_1 and the area above the dashed line A_2. They are determined by the following integrals:

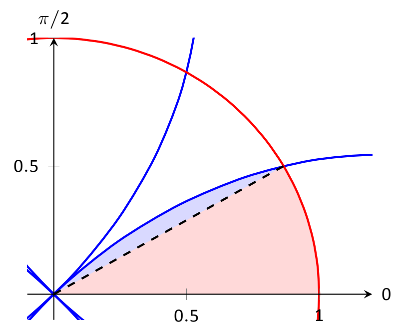
$$A_1=\frac{1}{2}\int_0^{\pi/6}(1)^2\,d\theta \qquad A_2=\frac{1}{2}\int_{\pi/6}^{\pi/4}\big(2\cos(2\theta)\big)^2\,d\theta.$$

(The upper bound of the integral computing A_2 is $\pi/4$ as $r=2\cos(2\theta)$ is at the pole when $\theta=\pi/4$.)

We omit the integration details and let the reader verify that $A_1=\pi/12$ and $A_2=\pi/12-\sqrt{3}/8$; the total area is $A=\pi/6-\sqrt{3}/8$.



(a)



(b)

Figure 9.5.7: Graphing the region bounded by the functions in Example 9.5.6.

Arc Length

As we have already considered the arc length of curves defined by rectangular and parametric equations, we now consider it in the context of polar equations. Recall that the arc length L of the graph defined by the parametric equations $x=f(t)$, $y=g(t)$ on $[a,b]$ is

$$L=\int_a^b\sqrt{f'(t)^2+g'(t)^2}\,dt=\int_a^b\sqrt{x'(t)^2+y'(t)^2}\,dt. \tag{9.1}$$

Now consider the polar function $r=f(\theta)$. We again use the identities $x=f(\theta)\cos\theta$ and $y=f(\theta)\sin\theta$ to create parametric equations based on the polar function. We compute $x'(\theta)$ and $y'(\theta)$ as done before when computing $\frac{dy}{dx}$, then apply Equation (9.1).

The expression $x'(\theta)^2 + y'(\theta)^2$ can be simplified a great deal; we leave this as an exercise and state that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

This leads us to the arc length formula.

Theorem 9.5.2 Arc Length of Polar Curves

Let $r = f(\theta)$ be a polar function with f' continuous on $[\alpha, \beta]$, on which the graph traces itself only once. The arc length L of the graph on $[\alpha, \beta]$ is

$$L = \int_\alpha^\beta \sqrt{f'(\theta)^2 + f(\theta)^2}\, d\theta = \int_\alpha^\beta \sqrt{(r')^2 + r^2}\, d\theta.$$

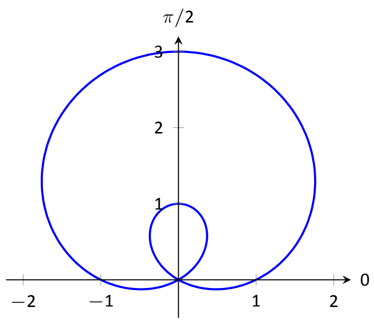


Figure 9.5.8: The limaçon in Example 9.5.7 whose arc length is measured.

Example 9.5.7 Arc length of a limaçon

Find the arc length of the limaçon $r = 1 + 2\sin t$.

Solution With $r = 1 + 2\sin t$, we have $r' = 2\cos t$. The limaçon is traced out once on $[0, 2\pi]$, giving us our bounds of integration. Applying Theorem 9.5.2, we have

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{(2\cos\theta)^2 + (1 + 2\sin\theta)^2}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\cos^2\theta + 4\sin^2\theta + 4\sin\theta + 1}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin\theta + 5}\, d\theta \\ &\approx 13.3649. \end{aligned}$$

The final integral cannot be solved in terms of elementary functions, so we resorted to a numerical approximation. (Simpson's Rule, with $n = 4$, approximates the value with 13.0608. Using $n = 22$ gives the value above, which is accurate to 4 places after the decimal.)

Surface Area

The formula for arc length leads us to a formula for surface area. The following Theorem is based on Theorem 9.3.2.

Theorem 9.5.3 Surface Area of a Solid of Revolution

Consider the graph of the polar equation $r = f(\theta)$, where f' is continuous on $[\alpha, \beta]$, on which the graph does not cross itself.

1. The surface area of the solid formed by revolving the graph about the initial ray ($\theta = 0$) is:
$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta)\sin\theta\sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta.$$
2. The surface area of the solid formed by revolving the graph about the line $\theta = \pi/2$ is:
$$\text{Surface Area} = 2\pi \int_\alpha^\beta f(\theta)\cos\theta\sqrt{f'(\theta)^2 + f(\theta)^2}\,d\theta.$$

Example 9.5.8 Surface area determined by a polar curve

Find the surface area formed by revolving one petal of the rose curve $r = \cos(2\theta)$ about its central axis (see Figure 9.5.9).

Solution We choose, as implied by the figure, to revolve the portion of the curve that lies on $[0, \pi/4]$ about the initial ray. Using Theorem 9.5.3 and the fact that $f'(\theta) = -2\sin(2\theta)$, we have

$$\begin{aligned}\text{Surface Area} &= 2\pi \int_0^{\pi/4} \cos(2\theta)\sin(\theta)\sqrt{\big(-2\sin(2\theta)\big)^2 + \big(\cos(2\theta)\big)^2}\,d\theta \\ &\approx 1.36707.\end{aligned}$$

The integral is another that cannot be evaluated in terms of elementary functions. Simpson's Rule, with $n = 4$, approximates the value at 1.36751.

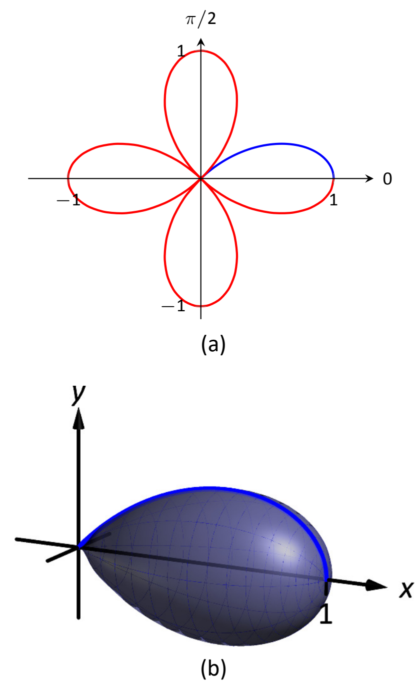


Figure 9.5.9: Finding the surface area of a rose–curve petal that is revolved around its central axis.

This chapter has been about curves in the plane. While there is great mathematics to be discovered in the two dimensions of a plane, we live in a three dimensional world and hence we should also look to do mathematics in 3D – that is, in *space*. The next chapter begins our exploration into space by introducing the topic of *vectors*, which are incredibly useful and powerful mathematical objects.

Exercises 9.5

Terms and Concepts

1. Given polar equation $r = f(\theta)$, how can one create parametric equations of the same curve?

2. With rectangular coordinates, it is natural to approximate area with ________; with polar coordinates, it is natural to approximate area with ________.

Problems

In Exercises 3 – 10, find:

(a) $\frac{dy}{dx}$

(b) the equation of the tangent and normal lines to the curve at the indicated θ–value.

3. $r = 1; \quad \theta = \pi/4$

4. $r = \cos\theta; \quad \theta = \pi/4$

5. $r = 1 + \sin\theta; \quad \theta = \pi/6$

6. $r = 1 - 3\cos\theta; \quad \theta = 3\pi/4$

7. $r = \theta; \quad \theta = \pi/2$

8. $r = \cos(3\theta); \quad \theta = \pi/6$

9. $r = \sin(4\theta); \quad \theta = \pi/3$

10. $r = \dfrac{1}{\sin\theta - \cos\theta}; \quad \theta = \pi$

In Exercises 11 – 14, find the values of θ in the given interval where the graph of the polar function has horizontal and vertical tangent lines.

11. $r = 3; \quad [0, 2\pi]$

12. $r = 2\sin\theta; \quad [0, \pi]$

13. $r = \cos(2\theta); \quad [0, 2\pi]$

14. $r = 1 + \cos\theta; \quad [0, 2\pi]$

In Exercises 15 – 16, find the equation of the lines tangent to the graph at the pole.

15. $r = \sin\theta; \quad [0, \pi]$

16. $r = \sin(3\theta); \quad [0, \pi]$

In Exercises 17 – 28, find the area of the described region.

17. Enclosed by the circle: $r = 4\sin\theta$

18. Enclosed by the circle $r = 5$

19. Enclosed by one petal of $r = \sin(3\theta)$

20. Enclosed by one petal of the rose curve $r = \cos(n\,\theta)$, where n is a positive integer.

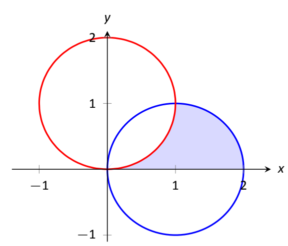
21. Enclosed by the cardioid $r = 1 - \sin\theta$

22. Enclosed by the inner loop of the limaçon $r = 1 + 2\cos\theta$

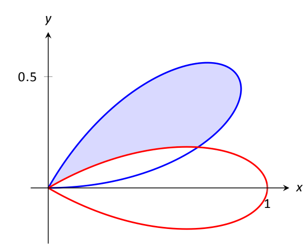
23. Enclosed by the outer loop of the limaçon $r = 1 + 2\cos\theta$ (including area enclosed by the inner loop)

24. Enclosed between the inner and outer loop of the limaçon $r = 1 + 2\cos\theta$

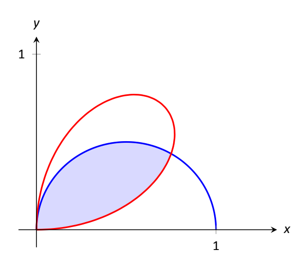
25. Enclosed by $r = 2\cos\theta$ and $r = 2\sin\theta$, as shown:



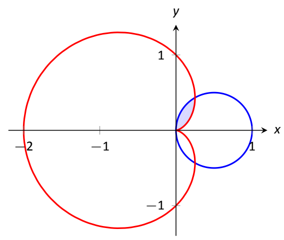
26. Enclosed by $r = \cos(3\theta)$ and $r = \sin(3\theta)$, as shown:



27. Enclosed by $r = \cos\theta$ and $r = \sin(2\theta)$, as shown:



28. Enclosed by $r = \cos\theta$ and $r = 1 - \cos\theta$, as shown:



34. Let $x(\theta) = f(\theta)\cos\theta$ and $y(\theta) = f(\theta)\sin\theta$. Show, as suggested by the text, that

$$x'(\theta)^2 + y'(\theta)^2 = f'(\theta)^2 + f(\theta)^2.$$

In Exercises 29 – 34, answer the questions involving arc length.

29. Use the arc length formula to compute the arc length of the circle $r = 2$.

30. Use the arc length formula to compute the arc length of the circle $r = 4\sin\theta$.

31. Use the arc length formula to compute the arc length of $r = \cos\theta + \sin\theta$.

32. Use the arc length formula to compute the arc length of the cardioid $r = 1 + \cos\theta$. (Hint: apply the formula, simplify, then use a Power–Reducing Formula to convert $1 + \cos\theta$ into a square.)

33. Approximate the arc length of one petal of the rose curve $r = \sin(3\theta)$ with Simpson's Rule and $n = 4$.

In Exercises 35 – 40, answer the questions involving surface area.

35. Find the surface area of the sphere formed by revolving the circle $r = 2$ about the initial ray.

36. Find the surface area of the sphere formed by revolving the circle $r = 2\cos\theta$ about the initial ray.

37. Find the surface area of the solid formed by revolving the cardioid $r = 1 + \cos\theta$ about the initial ray.

38. Find the surface area of the solid formed by revolving the circle $r = 2\cos\theta$ about the line $\theta = \pi/2$.

39. Find the surface area of the solid formed by revolving the line $r = 3\sec\theta$, $-\pi/4 \leq \theta \leq \pi/4$, about the line $\theta = \pi/2$.

40. Find the surface area of the solid formed by revolving the line $r = 3\sec\theta$, $0 \leq \theta \leq \pi/4$, about the initial ray.

A: Solutions To Selected Problems

Chapter 6

Section 6.1

1. Chain Rule.
3. $\frac{1}{8}(x^3-5)^8+C$
5. $\frac{1}{18}\left(x^2+1\right)^9+C$
7. $\frac{1}{2}\ln|2x+7|+C$
9. $\frac{2}{3}(x+3)^{3/2}-6(x+3)^{1/2}+C=\frac{2}{3}(x-6)\sqrt{x+3}+C$
11. $2e^{\sqrt{x}}+C$
13. $-\frac{1}{2x^2}-\frac{1}{x}+C$
15. $\frac{\sin^3(x)}{3}+C$
17. $-\frac{1}{6}\sin(3-6x)+C$
19. $\frac{1}{2}\ln|\sec(2x)+\tan(2x)|+C$
21. $\frac{\sin\left(x^2\right)}{2}+C$
23. The key is to rewrite $\cot x$ as $\cos x/\sin x$, and let $u=\sin x$.
25. $\frac{1}{3}e^{3x-1}+C$
27. $\frac{1}{2}e^{(x-1)^2}+C$
29. $\ln\left(e^x+1\right)+C$
31. $\frac{27^x}{\ln 27}+C$
33. $\frac{1}{2}\ln^2(x)+C$
35. $\frac{3}{2}(\ln x)^2+C$
37. $\frac{x^2}{2}+3x+\ln|x|+C$
39. $\frac{x^3}{3}-\frac{x^2}{2}+x-2\ln|x+1|+C$
41. $\frac{3}{2}x^2-8x+15\ln|x+1|+C$
43. $\sqrt{7}\tan^{-1}\left(\frac{x}{\sqrt{7}}\right)+C$
45. $14\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)+C$
47. $\frac{5}{4}\sec^{-1}(|x|/4)+C$
49. $\frac{\tan^{-1}\left(\frac{x-1}{\sqrt{7}}\right)}{\sqrt{7}}+C$
51. $3\sin^{-1}\left(\frac{x-4}{5}\right)+C$
53. $-\frac{1}{3(x^3+3)}+C$
55. $-\sqrt{1-x^2}+C$
57. $-\frac{2}{3}\cos^{\frac{3}{2}}(x)+C$
59. $\ln|x-5|+C$
61. $\frac{3x^2}{2}+\ln\left|x^2+3x+5\right|-5x+C$
63. $3\ln\left|3x^2+9x+7\right|+C$
65. $\frac{1}{18}\tan^{-1}\left(\frac{x^2}{9}\right)+C$
67. $\sec^{-1}(|2x|)+C$
69. $\frac{3}{2}\ln\left|x^2-2x+10\right|+\frac{1}{3}\tan^{-1}\left(\frac{x-1}{3}\right)+C$
71. $\frac{15}{2}\ln\left|x^2-10x+32\right|+x+\frac{41\tan^{-1}\left(\frac{x-5}{\sqrt{7}}\right)}{\sqrt{7}}+C$
73. $\frac{x^2}{2}+3\ln\left|x^2+4x+9\right|-4x+\frac{24\tan^{-1}\left(\frac{x+2}{\sqrt{5}}\right)}{\sqrt{5}}+C$
75. $\tan^{-1}(\sin(x))+C$
77. $3\sqrt{x^2-2x-6}+C$
79. $-\ln 2$
81. $2/3$
83. $(1-e)/2$
85. $\pi/2$

Section 6.2

1. T
3. Determining which functions in the integrand to set equal to "u" and which to set equal to "dv".
5. $\sin x-x\cos x+C$
7. $-x^2\cos x+2x\sin x+2\cos x+C$
9. $1/2e^{x^2}+C$
11. $-\frac{1}{2}xe^{-2x}-\frac{e^{-2x}}{4}+C$
13. $1/5e^{2x}(\sin x+2\cos x)+C$
15. $1/10e^{5x}(\sin(5x)+\cos(5x))+C$
17. $\sqrt{1-x^2}+x\sin^{-1}(x)+C$
19. $\frac{1}{2}x^2\tan^{-1}(x)-\frac{x}{2}+\frac{1}{2}\tan^{-1}(x)+C$
21. $\frac{1}{2}x^2\ln|x|-\frac{x^2}{4}+C$
23. $-\frac{x^2}{4}+\frac{1}{2}x^2\ln|x-1|-\frac{x}{2}-\frac{1}{2}\ln|x-1|+C$
25. $\frac{1}{3}x^3\ln|x|-\frac{x^3}{9}+C$
27. $(x+1)(\ln(x+1))^2-2(x+1)\ln(x+1)+2(x+1)+C$
29. $\ln|\sin(x)|-x\cot(x)+C$
31. $\frac{1}{3}(x^2-2)^{3/2}+C$
33. $x\sec x-\ln|\sec x+\tan x|+C$
35. $1/2x\big(\sin(\ln x)-\cos(\ln x)\big)+C$
37. $2\sin\left(\sqrt{x}\right)-2\sqrt{x}\cos\left(\sqrt{x}\right)+C$
39. $2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C$
41. π
43. 0
45. $1/2$
47. $\frac{3}{4e^2}-\frac{5}{4e^4}$
49. $1/5\left(e^{\pi}+e^{-\pi}\right)$

Section 6.3

1. F
3. F
5. $-\frac{1}{5}\cos^5(x)+C$
7. $\frac{1}{5}\cos^5 x-\frac{1}{3}\cos^3 x+C$
9. $\frac{1}{11}\sin^{11}x-\frac{2}{9}\sin^9 x+\frac{1}{7}\sin^7 x+C$
11. $\frac{x}{8}-\frac{1}{32}\sin(4x)+C$
13. $\frac{1}{2}\left(-\frac{1}{8}\cos(8x)-\frac{1}{2}\cos(2x)\right)+C$

15. $\frac{1}{2}\left(\frac{1}{4}\sin(4x) - \frac{1}{10}\sin(10x)\right) + C$

17. $\frac{1}{2}\left(\sin(x) + \frac{1}{3}\sin(3x)\right) + C$

19. $\frac{\tan^5(x)}{5} + C$

21. $\frac{\tan^6(x)}{6} + \frac{\tan^4(x)}{4} + C$

23. $\frac{\sec^5(x)}{5} - \frac{\sec^3(x)}{3} + C$

25. $\frac{1}{3}\tan^3 x - \tan x + x + C$

27. $\frac{1}{2}\left(\sec x\tan x - \ln|\sec x + \tan x|\right) + C$

29. $\frac{2}{5}$

31. $32/315$

33. $2/3$

35. $16/15$

Section 6.4

1. backwards

3. (a) $\tan^2\theta + 1 = \sec^2\theta$

   (b) $9\sec^2\theta$.

5. $\frac{1}{2}\left(x\sqrt{x^2+1} + \ln|\sqrt{x^2+1} + x|\right) + C$

7. $\frac{1}{2}\left(\sin^{-1}x + x\sqrt{1-x^2}\right) + C$

9. $\frac{1}{2}x\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2-1}| + C$

11. $x\sqrt{x^2+1/4} + \frac{1}{4}\ln|2\sqrt{x^2+1/4} + 2x| + C =$
$\frac{1}{2}x\sqrt{4x^2+1} + \frac{1}{4}\ln|\sqrt{4x^2+1} + 2x| + C$

13. $4\left(\frac{1}{2}x\sqrt{x^2-1/16} - \frac{1}{32}\ln|4x + 4\sqrt{x^2-1/16}|\right) + C =$
$\frac{1}{2}x\sqrt{16x^2-1} - \frac{1}{8}\ln|4x + \sqrt{16x^2-1}| + C$

15. $3\sin^{-1}\left(\frac{x}{\sqrt{7}}\right) + C$ (Trig. Subst. is not needed)

17. $\sqrt{x^2-11} - \sqrt{11}\sec^{-1}(x/\sqrt{11}) + C$

19. $\sqrt{x^2-3} + C$ (Trig. Subst. is not needed)

21. $-\frac{1}{\sqrt{x^2+9}} + C$ (Trig. Subst. is not needed)

23. $\frac{1}{18}\frac{x+2}{x^2+4x+13} + \frac{1}{54}\tan^{-1}\left(\frac{x+2}{2}\right) + C$

25. $\frac{1}{7}\left(-\frac{\sqrt{5-x^2}}{x} - \sin^{-1}(x/\sqrt{5})\right) + C$

27. $\pi/2$

29. $2\sqrt{2} + 2\ln(1+\sqrt{2})$

31. $9\sin^{-1}(1/3) + \sqrt{8}$ Note: the new lower bound is $\theta = \sin^{-1}(-1/3)$ and the new upper bound is $\theta = \sin^{-1}(1/3)$. The final answer comes with recognizing that $\sin^{-1}(-1/3) = -\sin^{-1}(1/3)$ and that $\cos\left(\sin^{-1}(1/3)\right) = \cos\left(\sin^{-1}(-1/3)\right) = \sqrt{8}/3$.

Section 6.5

1. rational

3. $\frac{A}{x} + \frac{B}{x-3}$

5. $\frac{A}{x-\sqrt{7}} + \frac{B}{x+\sqrt{7}}$

7. $3\ln|x-2| + 4\ln|x+5| + C$

9. $\frac{1}{3}(\ln|x+2| - \ln|x-2|) + C$

11. $\ln|x+5| - \frac{2}{x+5} + C$

13. $\frac{5}{x+1} + 7\ln|x| + 2\ln|x+1| + C$

15. $-\frac{1}{5}\ln|5x-1| + \frac{2}{3}\ln|3x-1| + \frac{3}{7}\ln|7x+3| + C$

17. $\frac{x^2}{2} + x + \frac{125}{9}\ln|x-5| + \frac{64}{9}\ln|x+4| - \frac{35}{2} + C$

19. $\frac{1}{6}\left(-\ln\left|x^2+2x+3\right| + 2\ln|x| - \sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)\right) + C$

21. $\ln\left|3x^2+5x-1\right| + 2\ln|x+1| + C$

23. $\frac{9}{10}\ln\left|x^2+9\right| + \frac{1}{5}\ln|x+1| - \frac{4}{15}\tan^{-1}\left(\frac{x}{3}\right) + C$

25. $3\left(\ln\left|x^2-2x+11\right| + \ln|x-9|\right) + 3\sqrt{\frac{2}{5}}\tan^{-1}\left(\frac{x-1}{\sqrt{10}}\right) + C$

27. $\ln(2000/243) \approx 2.108$

29. $-\pi/4 + \tan^{-1}3 - \ln(11/9) \approx 0.263$

Section 6.6

1. Because $\cosh x$ is always positive.

3.
$$\begin{aligned}\coth^2 x - \operatorname{csch}^2 x &= \left(\frac{e^x+e^{-x}}{e^x-e^{-x}}\right)^2 - \left(\frac{2}{e^x-e^{-x}}\right)^2\\ &= \frac{(e^{2x}+2+e^{-2x})-(4)}{e^{2x}-2+e^{-2x}}\\ &= \frac{e^{2x}-2+e^{-2x}}{e^{2x}-2+e^{-2x}}\\ &= 1\end{aligned}$$

5.
$$\begin{aligned}\cosh^2 x &= \left(\frac{e^x+e^{-x}}{2}\right)^2\\ &= \frac{e^{2x}+2+e^{-2x}}{4}\\ &= \frac{1}{2}\frac{(e^{2x}+e^{-2x})+2}{2}\\ &= \frac{1}{2}\left(\frac{e^{2x}+e^{-2x}}{2}+1\right)\\ &= \frac{\cosh 2x+1}{2}.\end{aligned}$$

7.
$$\begin{aligned}\frac{d}{dx}\left[\operatorname{sech} x\right] &= \frac{d}{dx}\left[\frac{2}{e^x+e^{-x}}\right]\\ &= \frac{-2(e^x-e^{-x})}{(e^x+e^{-x})^2}\\ &= -\frac{2(e^x-e^{-x})}{(e^x+e^{-x})(e^x+e^{-x})}\\ &= -\frac{2}{e^x+e^{-x}}\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}\\ &= -\operatorname{sech} x\tanh x\end{aligned}$$

9. $\displaystyle\int \tanh x\,dx = \int \frac{\sinh x}{\cosh x}\,dx$
Let $u = \cosh x$; $du = (\sinh x)dx$
$$\begin{aligned}&= \int \frac{1}{u}\,du\\ &= \ln|u| + C\\ &= \ln(\cosh x) + C.\end{aligned}$$

11. $2\cosh 2x$

13. $2x\sec^2(x^2)$

15. $\sinh^2 x + \cosh^2 x$

17. $\frac{-2x}{(x^2)\sqrt{1-x^4}}$

19. $\frac{4x}{\sqrt{4x^4-1}}$

21. $-\csc x$

23. $y = x$

25. $y = \frac{9}{25}(x + \ln 3) - \frac{4}{5}$

27. $y = x$

29. $1/2 \ln(\cosh(2x)) + C$

31. $1/2 \sinh^2 x + C$ or $1/2 \cosh^2 x + C$

33. $x \cosh(x) - \sinh(x) + C$

35. $\cosh^{-1} x/3 + C = \ln\left(x + \sqrt{x^2 - 9}\right) + C$

37. $\cosh^{-1}(x^2/2) + C = \ln(x^2 + \sqrt{x^4 - 4}) + C$

39. $\frac{1}{16} \tan^{-1}(x/2) + \frac{1}{32} \ln|x - 2| + \frac{1}{32} \ln|x + 2| + C$

41. $\tan^{-1}(e^x) + C$

43. $x \tanh^{-1} x + 1/2 \ln|x^2 - 1| + C$

45. 0

47. 2

Section 6.7

1. $0/0, \infty/\infty, 0 \cdot \infty, \infty - \infty, 0^0, 1^\infty, \infty^0$

3. F

5. derivatives; limits

7. Answers will vary.

9. 3

11. -1

13. 5

15. $2/3$

17. ∞

19. 0

21. 0

23. ∞

25. 0

27. -2

29. 0

31. 0

33. ∞

35. ∞

37. 0

39. 1

41. 1

43. 1

45. 1

47. 1

49. 2

51. $-\infty$

53. 0

Section 6.8

1. The interval of integration is finite, and the integrand is continuous on that interval.

3. converges; could also state < 10.

5. $p > 1$

7. $e^5/2$

9. $1/3$

11. $1/\ln 2$

13. diverges

15. 1

17. diverges

19. diverges

21. diverges

23. 1

25. 0

27. $-1/4$

29. diverges

31. 1

33. $1/2$

35. diverges; Limit Comparison Test with $1/x$.

37. diverges; Limit Comparison Test with $1/x$.

39. converges; Direct Comparison Test with e^{-x}.

41. converges; Direct Comparison Test with $1/(x^2 - 1)$.

43. converges; Direct Comparison Test with $1/e^x$.

Chapter 7

Section 7.1

1. T

3. Answers will vary.

5. $4\pi + \pi^2 \approx 22.436$

7. π

9. $1/2$

11. $1/\ln 4$

13. 4.5

15. $2 - \pi/2$

17. $1/6$

19. All enclosed regions have the same area, with regions being the reflection of adjacent regions. One region is formed on $[\pi/4, 5\pi/4]$, with area $2\sqrt{2}$.

21. On regions such as $[\pi/6, 5\pi/6]$, the area is $3\sqrt{3}/2$. On regions such as $[-\pi/2, \pi/6]$, the area is $3\sqrt{3}/4$.

23. $5/3$

25. $9/4$

27. $4/3$

29. 5

31. 133/20

Section 7.2

1. T

3. Recall that "dx" does not just "sit there;" it is multiplied by $A(x)$ and represents the thickness of a small slice of the solid. Therefore dx has units of in, giving $A(x)\,dx$ the units of in^3.

5. $48\pi\sqrt{3}/5$ units^3

7. $\pi^2/4$ units^3

9. $9\pi/2$ units^3

11. $\pi^2 - 2\pi$ units^3

13. (a) $\pi/2$

(b) $5\pi/6$

(c) $4\pi/5$

(d) $8\pi/15$

15. (a) $4\pi/3$

(b) $2\pi/3$

(c) $4\pi/3$

(d) $\pi/3$

17. (a) $\pi^2/2$

(b) $\pi^2/2 - 4\pi\sinh^{-1}(1)$

(c) $\pi^2/2 + 4\pi\sinh^{-1}(1)$

19. Placing the tip of the cone at the origin such that the x-axis runs through the center of the circular base, we have $A(x) = \pi x^2/4$. Thus the volume is $250\pi/3$ units^3.

21. Orient the cone such that the tip is at the origin and the x-axis is perpendicular to the base. The cross–sections of this cone are right, isosceles triangles with side length $2x/5$; thus the cross–sectional areas are $A(x) = 2x^2/25$, giving a volume of $80/3$ units^3.

Section 7.3

1. T
3. F
5. $9\pi/2$ units^3
7. $\pi^2 - 2\pi$ units^3
9. $48\pi\sqrt{3}/5$ units^3
11. $\pi^2/4$ units^3
13. (a) $4\pi/5$

(b) $8\pi/15$

(c) $\pi/2$

(d) $5\pi/6$

15. (a) $4\pi/3$

(b) $\pi/3$

(c) $4\pi/3$

(d) $2\pi/3$

17. (a) $2\pi(\sqrt{2}-1)$

(b) $2\pi(1-\sqrt{2}+\sinh^{-1}(1))$

Section 7.4

1. T
3. $\sqrt{2}$
5. $4/3$
7. $109/2$
9. $12/5$
11. $-\ln(2-\sqrt{3}) \approx 1.31696$
13. $\int_0^1 \sqrt{1+4x^2}\,dx$
15. $\int_0^1 \sqrt{1+\frac{1}{4x}}\,dx$
17. $\int_{-1}^1 \sqrt{1+\frac{x^2}{1-x^2}}\,dx$
19. $\int_1^2 \sqrt{1+\frac{1}{x^4}}\,dx$
21. 1.4790
23. Simpson's Rule fails, as it requires one to divide by 0. However, recognize the answer should be the same as for $y = x^2$; why?
25. Simpson's Rule fails.
27. 1.4058
29. $2\pi\int_0^1 2x\sqrt{5}\,dx = 2\pi\sqrt{5}$
31. $2\pi\int_0^1 x^3\sqrt{1+9x^4}\,dx = \pi/27(10\sqrt{10}-1)$
33. $2\pi\int_0^1 \sqrt{1-x^2}\sqrt{1+x/(1-x^2)}\,dx = 4\pi$

Section 7.5

1. In SI units, it is one joule, i.e., one Newton–metre, or kg·m/s^2·m. In Imperial Units, it is ft–lb.
3. Smaller.
5. (a) 500 ft–lb

(b) $100 - 50\sqrt{2} \approx 29.29$ ft–lb

7. (a) $\frac{1}{2}\cdot d\cdot l^2$ ft–lb

(b) 75 %

(c) $\ell(1-\sqrt{2}/2) \approx 0.2929\ell$

9. (a) 756 ft–lb

(b) 60,000 ft–lb

(c) Yes, for the cable accounts for about 1% of the total work.

11. 575 ft–lb
13. 0.05 J
15. 5/3 ft–lb
17. $f\cdot d/2$ J
19. 5 ft–lb
21. (a) 52,929.6 ft–lb

(b) 18,525.3 ft–lb

(c) When 3.83 ft of water have been pumped from the tank, leaving about 2.17 ft in the tank.

23. 212,135 ft–lb
25. 187,214 ft–lb
27. 4,917,150 J

Section 7.6

1. Answers will vary.
3. 499.2 lb
5. 6739.2 lb
7. 3920.7 lb
9. 2496 lb
11. 602.59 lb
13. (a) 2340 lb

(b) 5625 lb

15. (a) 1597.44 lb

(b) 3840 lb

17. (a) 56.42 lb

(b) 135.62 lb

19. 5.1 ft

Chapter 8

Section 8.1

1. If $x = e^{4t}$, then $x' = 4e^{4t} = 4x$, $x'' = 16e^{4t} = 16x$, and $x''' = 64e^{4t} = 64x$. Thus $x''' - 12x'' + 48x' - 64x = 64x - 192x + 192x - 64x = 0$.

3. Yes: If $y = \sin t$ then $\dfrac{dy}{dt} = \cos t$ and $1 - \sin^2 t = \cos^2 t$.

5. Since $x(0) = Ce^0 = C$, we need $C = 100$. Verification is left to the student.

7. One option is $x(t) = 2\sin(t)$, since $x'(t) = 2\cos(t)$, and $(2\cos(t))^2 + (2\sin(t))^2 = 4$. There are other options.

9. Yes: any constant function will do the job.

11. Yes.

13. $C_1 = 100$, $C_2 = -90$

Section 8.2

1. $y = \dfrac{1}{3}x^3 + \dfrac{1}{2}x^2 + \dfrac{13}{6}$

3. Equivalent answers: $y = \frac{1}{2}(\ln(1-x) - \ln(1+x))$ or $y = -\tanh^{-1}(x)$

5. Assuming $y \neq \pm 1$, we can write $\dfrac{dx}{dy} = \dfrac{1}{y^2 - 1}$, which gives $x = -\tanh^{-1}(y) + C$, so $\tanh^{-1}(y) = C - x$, and thus $y = \tanh(C - x)$. The condition $y(0) = 3$ gives $\tanh(C) = 3$, so $C = \tanh^{-1}(3)$.

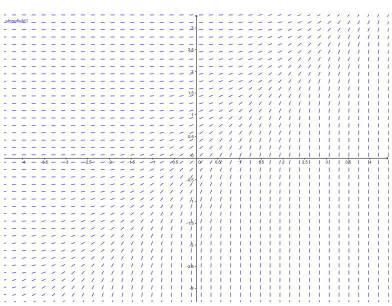
7. Integrating once gives $y' = -\cos x + c$, and $y'(0) = 2$ implies $c = 3$. Integrating again gives $y = -\sin x + 3x + d$, and since $y(0) = 0$, $d = 0$.

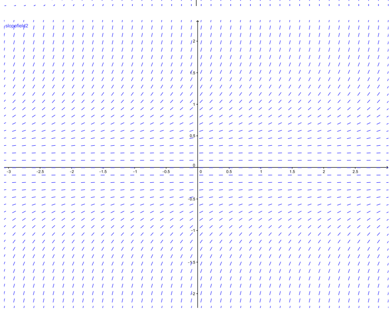
9. Integrating gives $x(t) = \int_0^t \sin(u^2)\,du + \frac{1}{2}t^2 + 20$.

11. $x = (3t - 2)^{1/3}$

13. 170

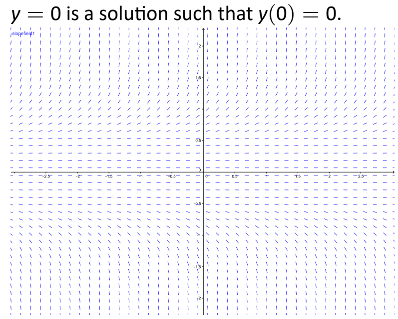
Section 8.3

1. 

3. 

5. Yes, on both accounts. For $f(x, y) = y\sqrt{|x|}$, the function is continuous everywhere, and the partial derivative $\dfrac{\partial f}{\partial y}(x, y) = \sqrt{|x|}$ exists and is everywhere continuous.

7. We have to have $y \to \infty$ as $x \to infty$: We begin at $(0, 0)$ and $f(0, 0) > 1$, so $y' > 1$. If we compare to the function $f(x) = x$, we have $y(0) = f(0)$ and $y'(x) > f'(x)$ for all x, which implies that $y(x) > f(x)$ for all x.

9. $y = 0$ is a solution such that $y(0) = 0$.


11. No, the equation is not defined at $(x, y) = (1, 0)$.

Section 8.4

1. $y^2 = x^2 + C$

3. $x = -\tanh(t^2/2)$

5. Notice that $xy + x + y + 1 = (x+1)(y+1)$. This gives $y = Ce^{x^2/2 + x} - 1$.

7. $y = \tan\left(\dfrac{\pi}{4} + \arctan x\right) = \dfrac{1 + x}{1 - x}$.

9. $y = \ln\left(\dfrac{x^2}{2} + e\right)$.

11. $y = \exp\left(\int_0^x e^{-t^2}\,dt\right)$.

13. $y = Ce^{x^2}$

15. $x^3 + x = t + 2$

17. $\sin(y) = -\cos(x) + C$

Section 8.5

In the exercises, feel free to leave answer as a definite integral if a closed form solution cannot be found. If you can find a closed form solution, you should give that.

1. The integrating factor is $r(x) = e^{x^2/2}$.
   The solution is $y = 1 + Ce^{-x^2/2}$.

3. The integrating factor is e^{x^3}.
   The solution is $y = e^{-x^3}(\sin x - x\cos x)$.

5. After rewriting as $y' + (x^2 + x)y = 3(x^2 + 1)$, we get the integrating factor $r(x) = e^{x^4/4 + x^2/2}$.
   The solution is $y = 3e^{-x^4/4 - x^2/2}\int_0^x (t^2 + 1)e^{t^4/4 + t^2/2}\,dt$. (There is no closed form solution.)

7. (a) The general solution is
$$x(t) = \frac{\omega A_0}{\omega^2 - k^2}\left(\sin(\omega t) + \frac{k}{\omega}\cos(\omega t)\right) + Ce^{-kt}.$$
   (We hope you haven't forgotten how to integrate by parts!)
   (b)They won't. Since $k > 0$, the term that is determined by the initial conditions decays exponentially, so for $t >> 0$, there won't be much of a contribution from this term.

9. $k = 9/8$ grams per litre.

11. $y = 2e^{\cos(2x) + 1} + 1$

13. $P(5) = 1000e^{2\times 5 - 0.05\times 5^2} = 1000e^{8.75} \approx 6.31 \times 10^6$

Section 8.6

1. $x(1) = 8.5$.

3. We get $y(1) \approx y_4 = 2.4414$.

5. Approximately: 1.0000, 1.2397, 1.3829

7. (a) 0, 0, 0
(b) $x = 0$ is a solution so errors are: 0, 0, 0.

Chapter 9

Section 9.1

1. When defining the conics as the intersections of a plane and a double napped cone, degenerate conics are created when the plane intersects the tips of the cones (usually taken as the origin). Nondegenerate conics are formed when this plane does not contain the origin.

3. Hyperbola

5. With a horizontal transverse axis, the x^2 term has a positive coefficient; with a vertical transverse axis, the y^2 term has a positive coefficient.

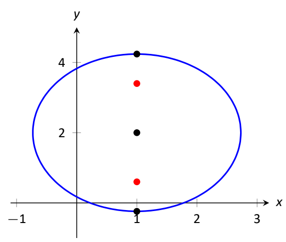
7. $y = \frac{1}{2}(x-3)^2 + \frac{3}{2}$

9. $x = -\frac{1}{4}(y-5)^2 + 2$

11. $y = -\frac{1}{4}(x-1)^2 + 2$

13. $y = 4x^2$

15. focus: $(0, 1)$; directrix: $y = -1$. The point P is 2 units from each.

17. 

19. $\frac{(x+1)^2}{9} + \frac{(y-2)^2}{4} = 1$; foci at $(-1 \pm \sqrt{5}, 2)$; $e = \sqrt{5}/3$

21. $\frac{x^2}{9} + \frac{y^2}{5} = 1$

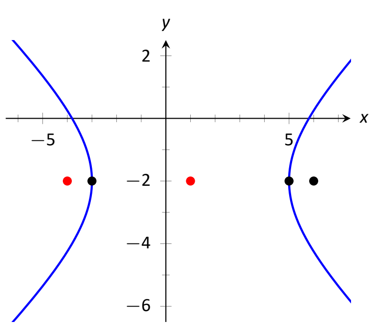
23. $\frac{(x-2)^2}{45} + \frac{y^2}{49} = 1$

25. $\frac{(x-1)^2}{2} + (y-2)^2 = 1$

27. $\frac{x^2}{4} + \frac{(y-3)^2}{6} = 1$

29. $x^2 - \frac{y^2}{3} = 1$

31. $\frac{(y-3)^2}{4} - \frac{(x-1)^2}{9} = 1$

33. 

35. $\frac{x^2}{4} - \frac{y^2}{5} = 1$

37. $\frac{(x-3)^2}{16} - \frac{(y-3)^2}{9} = 1$

39. $\frac{x^2}{4} - \frac{y^2}{3} = 1$

41. $(y-2)^2 - \frac{x^2}{10} = 1$

43. (a) $c = \sqrt{12-4} = 2\sqrt{2}$.

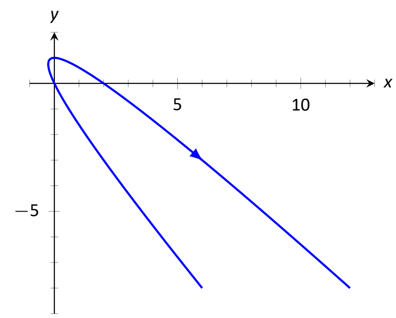
(b) The sum of distances for each point is $2\sqrt{12} \approx 6.9282$.

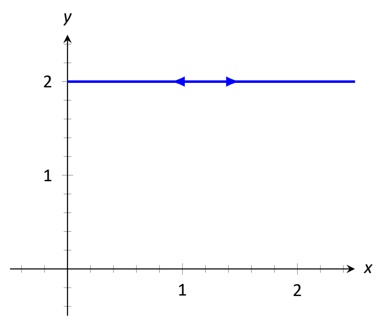
45. The sound originated from a point approximately 31m to the left of B and 1340m above it.

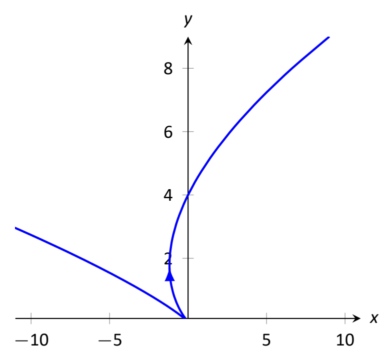
Section 9.2

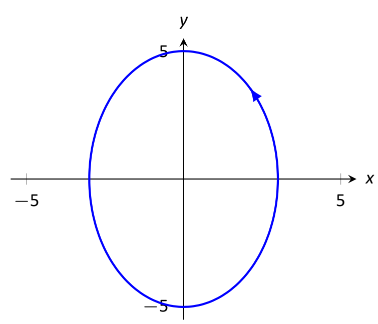
1. T

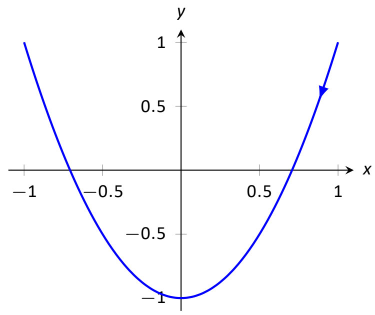
3. rectangular

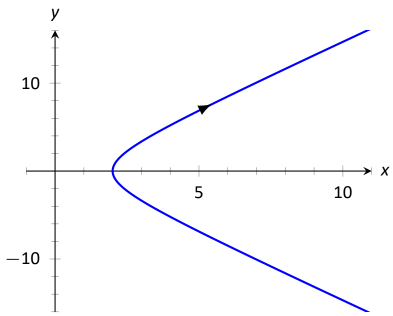
5. 

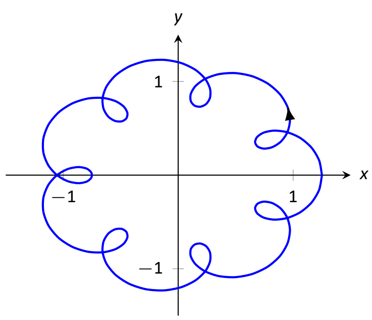
7. 

9. 

11. 

13. 

15. 

17. 

19. (a) Traces the parabola $y = x^2$, moves from left to right.
    (b) Traces the parabola $y = x^2$, but only from $-1 \leq x \leq 1$; traces this portion back and forth infinitely.
    (c) Traces the parabola $y = x^2$, but only for $0 < x$. Moves left to right.
    (d) Traces the parabola $y = x^2$, moves from right to left.

21. $y = -1.5x + 8.5$

23. $\frac{(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1$

25. $y = 2x + 3$

27. $y = e^{2x} - 1$

29. $x^2 - y^2 = 1$

31. $y = \frac{b}{a}(x - x_0) + y_0$; line through (x_0, y_0) with slope b/a.

33. $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$; ellipse centered at (h, k) with horizontal axis of length $2a$ and vertical axis of length $2b$.

35. $x = (t + 11)/6$, $y = (t^2 - 97)/12$. At $t = 1$, $x = 2$, $y = -8$.
    $y' = 6x - 11$; when $x = 2$, $y' = 1$.

37. $x = \cos^{-1} t$, $y = \sqrt{1 - t^2}$. At $t = 1$, $x = 0$, $y = 0$.
    $y' = \cos x$; when $x = 0$, $y' = 1$.

39. $t = \pm 1$

41. $t = \pi/2, 3\pi/2$

43. $t = -1$

45. $t = \ldots \pi/2,\ 3\pi/2,\ 5\pi/2,\ \ldots$

47. $x = 4t$, $y = -16t^2 + 64t$

49. $x = 10t$, $y = -16t^2 + 320t$

51. $x = 3\cos(2\pi t) + 1$, $y = 3\sin(2\pi t) + 1$; other answers possible

53. $x = 5\cos t$, $y = \sqrt{24}\sin t$; other answers possible

55. $x = 2\tan t$, $y = \pm 6\sec t$; other answers possible

Section 9.3

1. F

3. F

5. (a) $\frac{dy}{dx} = 2t$
   (b) Tangent line: $y = 2(x - 1) + 1$; normal line: $y = -1/2(x - 1) + 1$

7. (a) $\frac{dy}{dx} = \frac{2t+1}{2t-1}$
   (b) Tangent line: $y = 3x + 2$; normal line: $y = -1/3x + 2$

9. (a) $\frac{dy}{dx} = \csc t$
   (b) $t = \pi/4$: Tangent line: $y = \sqrt{2}(x - \sqrt{2}) + 1$; normal line: $y = -1/\sqrt{2}(x - \sqrt{2}) + 1$

11. (a) $\frac{dy}{dx} = \frac{\cos t \sin(2t) + \sin t \cos(2t)}{-\sin t \sin(2t) + 2\cos t \cos(2t)}$
    (b) Tangent line: $y = x - \sqrt{2}$; normal line: $y = -x - \sqrt{2}$

13. $t = 0$

15. $t = -1/2$

17. The graph does not have a horizontal tangent line.

19. The solution is non-trivial; use identities $\sin(2t) = 2\sin t\cos t$ and $\cos(2t) = \cos^2 t - \sin^2 t$ to rewrite $g'(t) = 2\sin t(2\cos^2 t - \sin^2 t)$. On $[0, 2\pi]$, $\sin t = 0$ when $t = 0, \pi, 2\pi$, and $2\cos^2 t - \sin^2 t = 0$ when $t = \tan^{-1}(\sqrt{2}),\ \pi \pm \tan^{-1}(\sqrt{2}),\ 2\pi - \tan^{-1}(\sqrt{2})$.

21. $t_0 = 0$; $\lim_{t\to 0} \frac{dy}{dx} = 0$.

23. $t_0 = 1$; $\lim_{t\to 1} \frac{dy}{dx} = \infty$.

25. $\frac{d^2y}{dx^2} = 2$; always concave up

27. $\frac{d^2y}{dx^2} = -\frac{4}{(2t-1)^3}$; concave up on $(-\infty, 1/2)$; concave down on $(1/2, \infty)$.

29. $\frac{d^2y}{dx^2} = -\cot^3 t$; concave up on $(-\infty, 0)$; concave down on $(0, \infty)$.

31. $\frac{d^2y}{dx^2} = \frac{4(13+3\cos(4t))}{(\cos t + 3\cos(3t))^3}$, obtained with a computer algebra system; concave up on $\big(-\tan^{-1}(\sqrt{2}/2), \tan^{-1}(\sqrt{2}/2)\big)$, concave down on $\big(-\pi/2, -\tan^{-1}(\sqrt{2}/2)\big) \cup \big(\tan^{-1}(\sqrt{2}/2), \pi/2\big)$

33. $L = 6\pi$

35. $L = 2\sqrt{34}$

37. $L \approx 2.4416$ (actual value: $L = 2.42211$)

39. $L \approx 4.19216$ (actual value: $L = 4.18308$)

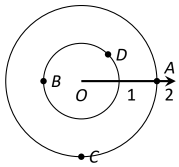
41. The answer is 16π for both (of course), but the integrals are different.

43. $SA \approx 8.50101$ (actual value $SA = 8.02851$)

Section 9.4

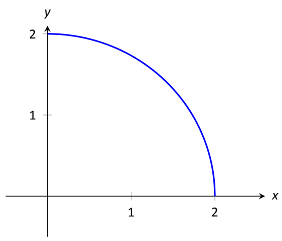
1. Answers will vary.

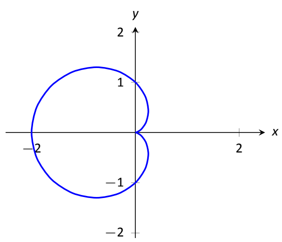
3. T

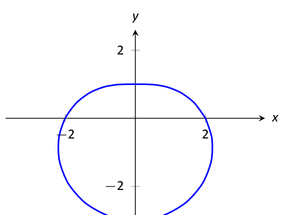
5. 

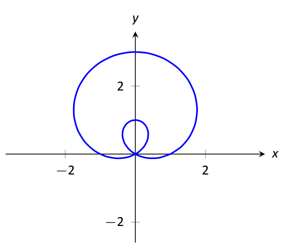
7. $A = P(2.5, \pi/4)$ and $P(-2.5, 5\pi/4)$;
$B = P(-1, 5\pi/6)$ and $P(1, 11\pi/6)$;
$C = P(3, 4\pi/3)$ and $P(-3, \pi/3)$;
$D = P(1.5, 2\pi/3)$ and $P(-1.5, 5\pi/3)$;

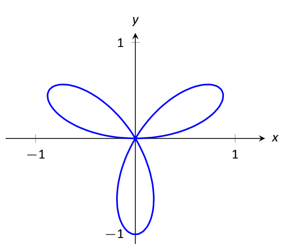
9. $A = (\sqrt{2}, \sqrt{2})$
$B = (\sqrt{2}, -\sqrt{2})$
$C = P(\sqrt{5}, -0.46)$
$D = P(\sqrt{5}, 2.68)$

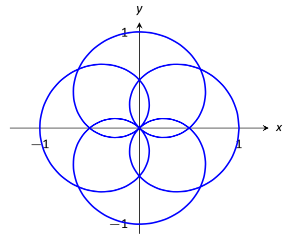
11. 

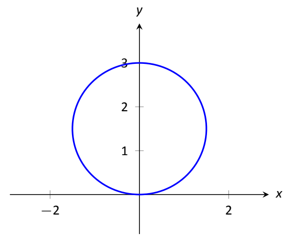
13. 

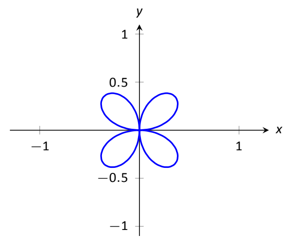
15. 

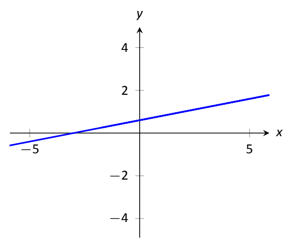
17. 

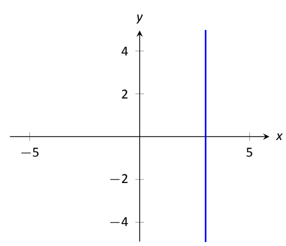
19. 

21. 

23. 

25. 

27. 

29. 

31. $(x-3)^2 + y^2 = 3$

33. $(x-1/2)^2 + (y-1/2)^2 = 1/2$

35. $x = 3$

37. $x^4 + x^2y^2 - y^2 = 0$

39. $x^2 + y^2 = 4$

41. $\theta = \pi/4$

43. $r = 5\sec\theta$

45. $r = \cos\theta/\sin^2\theta$

47. $r = \sqrt{7}$

49. $P(\sqrt{3}/2, \pi/6)$, $P(0, \pi/2)$, $P(-\sqrt{3}/2, 5\pi/6)$

51. $P(0,0) = P(0, \pi/2)$, $P(\sqrt{2}, \pi/4)$

53. $P(\sqrt{2}/2, \pi/12)$, $P(-\sqrt{2}/2, 5\pi/12)$, $P(\sqrt{2}/2, 3\pi/4)$

55. For all points, $r = 1$; $\theta =$
$\pi/12,\ 5\pi/12,\ 7\pi/12,\ 11\pi/12,\ 13\pi/12,\ 17\pi/12,\ 19\pi/12,\ 23\pi/12.$

57. Answers will vary. If m and n do not have any common factors, then an interval of $2n\pi$ is needed to sketch the entire graph.

Section 9.5

1. Using $x = r\cos\theta$ and $y = r\sin\theta$, we can write $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$.

3. (a) $\frac{dy}{dx} = -\cot\theta$
   (b) tangent line: $y = -(x - \sqrt{2}/2) + \sqrt{2}/2$; normal line: $y = x$

5. (a) $\frac{dy}{dx} = \frac{\cos\theta(1+2\sin\theta)}{\cos^2\theta - \sin\theta(1+\sin\theta)}$
   (b) tangent line: $x = 3\sqrt{3}/4$; normal line: $y = 3/4$

7. (a) $\frac{dy}{dx} = \frac{\theta\cos\theta + \sin\theta}{\cos\theta - \theta\sin\theta}$
   (b) tangent line: $y = -2/\pi x + \pi/2$; normal line: $y = \pi/2x + \pi/2$

9. (a) $\frac{dy}{dx} = \frac{4\sin(\theta)\cos(4\theta) + \sin(4\theta)\cos(\theta)}{4\cos(\theta)\cos(4\theta) - \sin(\theta)\sin(4\theta)}$
   (b) tangent line: $y = 5\sqrt{3}(x + \sqrt{3}/4) - 3/4$; normal line: $y = -1/5\sqrt{3}(x + \sqrt{3}/4) - 3/4$

11. horizontal: $\theta = \pi/2, 3\pi/2$;
    vertical: $\theta = 0, \pi, 2\pi$

13. horizontal: $\theta = \tan^{-1}(1/\sqrt{5}),\ \pi/2,\ \pi - \tan^{-1}(1/\sqrt{5}),\ \pi + \tan^{-1}(1/\sqrt{5}),\ 3\pi/2,\ 2\pi - \tan^{-1}(1/\sqrt{5})$;
    vertical: $\theta = 0,\ \tan^{-1}(\sqrt{5}),\ \pi - \tan^{-1}(\sqrt{5}),\ \pi,\ \pi + \tan^{-1}(\sqrt{5}),\ 2\pi - \tan^{-1}(\sqrt{5})$

15. In polar: $\theta = 0 \cong \theta = \pi$
    In rectangular: $y = 0$

17. area = 4π

19. area = $\pi/12$

21. area = $3\pi/2$

23. area = $2\pi + 3\sqrt{3}/2$

25. area = 1

27. area = $\frac{1}{32}(4\pi - 3\sqrt{3})$

29. 4π

31. area = $\sqrt{2}\pi$

33. $L \approx 2.2592$; (actual value $L = 2.22748$)

35. $SA = 16\pi$

37. $SA = 32\pi/5$

39. $SA = 36\pi$

Index

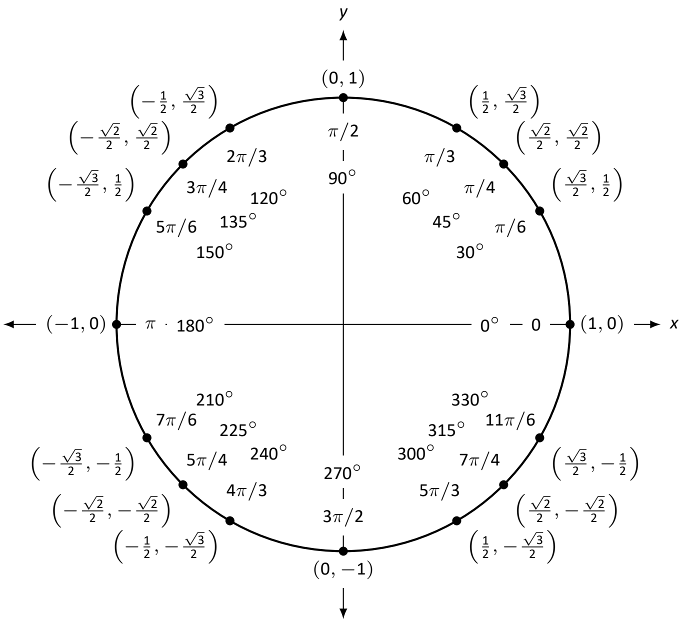
Differentiation Rules

1. $\frac{d}{dx}(cx) = c$
2. $\frac{d}{dx}(u \pm v) = u' \pm v'$
3. $\frac{d}{dx}(u \cdot v) = uv' + u'v$
4. $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu' - uv'}{v^2}$
5. $\frac{d}{dx}(u(v)) = u'(v)v'$
6. $\frac{d}{dx}(c) = 0$
7. $\frac{d}{dx}(x) = 1$
8. $\frac{d}{dx}(x^n) = nx^{n-1}$
9. $\frac{d}{dx}(e^x) = e^x$
10. $\frac{d}{dx}(a^x) = \ln a \cdot a^x$
11. $\frac{d}{dx}(\ln x) = \frac{1}{x}$
12. $\frac{d}{dx}(\log_a x) = \frac{1}{\ln a} \cdot \frac{1}{x}$
13. $\frac{d}{dx}(\sin x) = \cos x$
14. $\frac{d}{dx}(\cos x) = -\sin x$
15. $\frac{d}{dx}(\csc x) = -\csc x \cot x$
16. $\frac{d}{dx}(\sec x) = \sec x \tan x$
17. $\frac{d}{dx}(\tan x) = \sec^2 x$
18. $\frac{d}{dx}(\cot x) = -\csc^2 x$
19. $\frac{d}{dx}\left(\sin^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}$
20. $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$
21. $\frac{d}{dx}\left(\csc^{-1} x\right) = \frac{-1}{x\sqrt{x^2-1}}$
22. $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}}$
23. $\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{1+x^2}$
24. $\frac{d}{dx}\left(\cot^{-1} x\right) = \frac{-1}{1+x^2}$
25. $\frac{d}{dx}(\cosh x) = \sinh x$
26. $\frac{d}{dx}(\sinh x) = \cosh x$
27. $\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$
28. $\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$
29. $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$
30. $\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$
31. $\frac{d}{dx}\left(\cosh^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}$
32. $\frac{d}{dx}\left(\sinh^{-1} x\right) = \frac{1}{\sqrt{x^2+1}}$
33. $\frac{d}{dx}\left(\operatorname{sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}$
34. $\frac{d}{dx}\left(\operatorname{csch}^{-1} x\right) = \frac{-1}{|x|\sqrt{1+x^2}}$
35. $\frac{d}{dx}\left(\tanh^{-1} x\right) = \frac{1}{1-x^2}$
36. $\frac{d}{dx}\left(\coth^{-1} x\right) = \frac{1}{1-x^2}$

Integration Rules

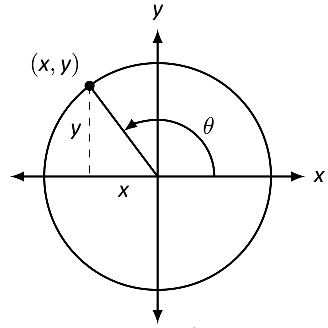
1. $\int c \cdot f(x)\, dx = c \int f(x)\, dx$
2. $\int f(x) \pm g(x)\, dx =$ $\int f(x)\, dx \pm \int g(x)\, dx$
3. $\int 0\, dx = C$
4. $\int 1\, dx = x + C$
5. $\int x^n\, dx = \frac{1}{n+1}x^{n+1} + C,\ n \neq -1$ $n \neq -1$
6. $\int e^x\, dx = e^x + C$
7. $\int a^x\, dx = \frac{1}{\ln a} \cdot a^x + C$
8. $\int \frac{1}{x}\, dx = \ln|x| + C$
9. $\int \cos x\, dx = \sin x + C$
10. $\int \sin x\, dx = -\cos x + C$
11. $\int \tan x\, dx = -\ln|\cos x| + C$
12. $\int \sec x\, dx = \ln|\sec x + \tan x| + C$
13. $\int \csc x\, dx = -\ln|\csc x + \cot x| + C$
14. $\int \cot x\, dx = \ln|\sin x| + C$
15. $\int \sec^2 x\, dx = \tan x + C$
16. $\int \csc^2 x\, dx = -\cot x + C$
17. $\int \sec x \tan x\, dx = \sec x + C$
18. $\int \csc x \cot x\, dx = -\csc x + C$
19. $\int \cos^2 x\, dx = \frac{1}{2}x + \frac{1}{4}\sin(2x) + C$
20. $\int \sin^2 x\, dx = \frac{1}{2}x - \frac{1}{4}\sin(2x) + C$
21. $\int \frac{1}{x^2+a^2}\, dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
22. $\int \frac{1}{\sqrt{a^2-x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C$
23. $\int \frac{1}{x\sqrt{x^2-a^2}}\, dx = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + C$
24. $\int \cosh x\, dx = \sinh x + C$
25. $\int \sinh x\, dx = \cosh x + C$
26. $\int \tanh x\, dx = \ln(\cosh x) + C$
27. $\int \coth x\, dx = \ln|\sinh x| + C$
28. $\int \frac{1}{\sqrt{x^2-a^2}}\, dx = \ln\left|x + \sqrt{x^2-a^2}\right| + C$
29. $\int \frac{1}{\sqrt{x^2+a^2}}\, dx = \ln\left|x + \sqrt{x^2+a^2}\right| + C$
30. $\int \frac{1}{a^2-x^2}\, dx = \frac{1}{2}\ln\left|\frac{a+x}{a-x}\right| + C$
31. $\int \frac{1}{x\sqrt{a^2-x^2}}\, dx = \frac{1}{a}\ln\left(\frac{x}{a+\sqrt{a^2-x^2}}\right) + C$
32. $\int \frac{1}{x\sqrt{x^2+a^2}}\, dx = \frac{1}{a}\ln\left|\frac{x}{a+\sqrt{x^2+a^2}}\right| + C$

The Unit Circle



Definitions of the Trigonometric Functions

Unit Circle Definition

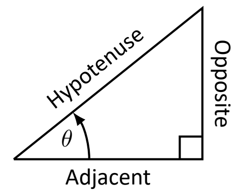


$$\sin\theta = y \qquad \cos\theta = x$$

$$\csc\theta = \frac{1}{y} \qquad \sec\theta = \frac{1}{x}$$

$$\tan\theta = \frac{y}{x} \qquad \cot\theta = \frac{x}{y}$$

Right Triangle Definition



$$\sin\theta = \frac{\text{O}}{\text{H}} \qquad \csc\theta = \frac{\text{H}}{\text{O}}$$

$$\cos\theta = \frac{\text{A}}{\text{H}} \qquad \sec\theta = \frac{\text{H}}{\text{A}}$$

$$\tan\theta = \frac{\text{O}}{\text{A}} \qquad \cot\theta = \frac{\text{A}}{\text{O}}$$

Common Trigonometric Identities

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

Cofunction Identities

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x \qquad \csc\left(\frac{\pi}{2} - x\right) = \sec x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x \qquad \sec\left(\frac{\pi}{2} - x\right) = \csc x$$

$$\tan\left(\frac{\pi}{2} - x\right) = \cot x \qquad \cot\left(\frac{\pi}{2} - x\right) = \tan x$$

Double Angle Formulas

$$\sin 2x = 2\sin x\cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

Sum to Product Formulas

$$\sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)$$

$$\cos x + \cos y = 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$$

$$\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$$

Power–Reducing Formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}$$

$$\cos^2 x = \frac{1 + \cos 2x}{2}$$

$$\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}$$

Even/Odd Identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

$$\tan(-x) = -\tan x$$

$$\csc(-x) = -\csc x$$

$$\sec(-x) = \sec x$$

$$\cot(-x) = -\cot x$$

Product to Sum Formulas

$$\sin x\sin y = \frac{1}{2}\big(\cos(x-y) - \cos(x+y)\big)$$

$$\cos x\cos y = \frac{1}{2}\big(\cos(x-y) + \cos(x+y)\big)$$

$$\sin x\cos y = \frac{1}{2}\big(\sin(x+y) + \sin(x-y)\big)$$

Angle Sum/Difference Formulas

$$\sin(x \pm y) = \sin x\cos y \pm \cos x\sin y$$

$$\cos(x \pm y) = \cos x\cos y \mp \sin x\sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x\tan y}$$

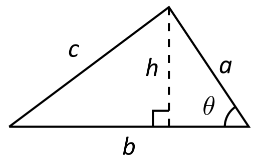
Areas and Volumes

Triangles

$h = a \sin\theta$

Area = $\frac{1}{2}bh$

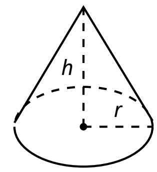
Law of Cosines:
$c^2 = a^2 + b^2 - 2ab\cos\theta$



Right Circular Cone

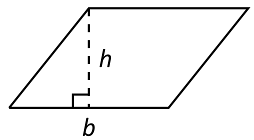
Volume = $\frac{1}{3}\pi r^2 h$

Surface Area =
$\pi r\sqrt{r^2 + h^2} + \pi r^2$



Parallelograms

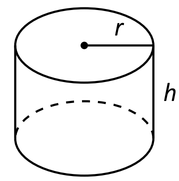
Area = bh



Right Circular Cylinder

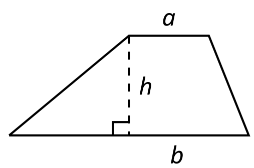
Volume = $\pi r^2 h$

Surface Area =
$2\pi rh + 2\pi r^2$



Trapezoids

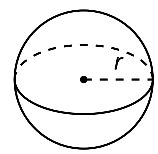
Area = $\frac{1}{2}(a + b)h$



Sphere

Volume = $\frac{4}{3}\pi r^3$

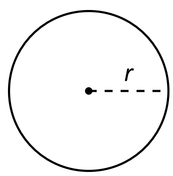
Surface Area =$4\pi r^2$



Circles

Area = πr^2

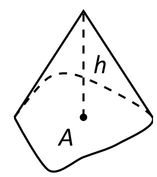
Circumference = $2\pi r$



General Cone

Area of Base = A

Volume = $\frac{1}{3}Ah$

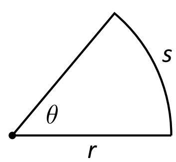


Sectors of Circles

θ in radians

Area = $\frac{1}{2}\theta r^2$

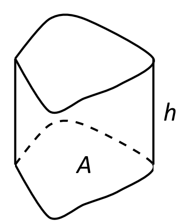
$s = r\theta$



General Right Cylinder

Area of Base = A

Volume = Ah



Algebra

Factors and Zeros of Polynomials

Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial. If $p(a) = 0$, then a is a *zero* of the polynomial and a solution of the equation $p(x) = 0$. Furthermore, $(x - a)$ is a *factor* of the polynomial.

Fundamental Theorem of Algebra

An nth degree polynomial has n (not necessarily distinct) zeros. Although all of these zeros may be imaginary, a real polynomial of odd degree must have at least one real zero.

Quadratic Formula

If $p(x) = ax^2 + bx + c$, and $0 \le b^2 - 4ac$, then the real zeros of p are $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Special Factors

$x^2 - a^2 = (x - a)(x + a)$

$x^3 - a^3 = (x - a)(x^2 + ax + a^2)$

$x^3 + a^3 = (x + a)(x^2 - ax + a^2)$

$x^4 - a^4 = (x^2 - a^2)(x^2 + a^2)$

$(x + y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \cdots + nxy^{n-1} + y^n$

$(x - y)^n = x^n - nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 - \cdots \pm nxy^{n-1} \mp y^n$

Binomial Theorem

$(x + y)^2 = x^2 + 2xy + y^2$

$(x - y)^2 = x^2 - 2xy + y^2$

$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$

$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

$(x - y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$

Rational Zero Theorem

If $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients, then every *rational zero* of p is of the form $x = r/s$, where r is a factor of a_0 and s is a factor of a_n.

Factoring by Grouping

$acx^3 + adx^2 + bcx + bd = ax^2(cx + d) + b(cx + d) = (ax^2 + b)(cx + d)$

Arithmetic Operations

$ab + ac = a(b + c)$

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

$$\frac{a + b}{c} = \frac{a}{c} + \frac{b}{c}$$

$$\frac{\left(\frac{a}{b}\right)}{\left(\frac{c}{d}\right)} = \left(\frac{a}{b}\right)\left(\frac{d}{c}\right) = \frac{ad}{bc}$$

$$\frac{\left(\frac{a}{b}\right)}{c} = \frac{a}{bc}$$

$$\frac{a}{\left(\frac{b}{c}\right)} = \frac{ac}{b}$$

$$a\left(\frac{b}{c}\right) = \frac{ab}{c}$$

$$\frac{a - b}{c - d} = \frac{b - a}{d - c}$$

$$\frac{ab + ac}{a} = b + c$$

Exponents and Radicals

$a^0 = 1, \quad a \neq 0$

$(ab)^x = a^x b^x$

$a^x a^y = a^{x+y}$

$\sqrt{a} = a^{1/2}$

$\frac{a^x}{a^y} = a^{x-y}$

$\sqrt[n]{a} = a^{1/n}$

$\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$

$\sqrt[n]{a^m} = a^{m/n}$

$a^{-x} = \frac{1}{a^x}$

$\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$

$(a^x)^y = a^{xy}$

$\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$

Additional Formulas

Summation Formulas:

$$\sum_{i=1}^{n} c = cn \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_1) + 2f(x_2) + 2f(x_3) + \ldots + 2f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^3}{12n^2}\left[\max\left|f''(x)\right|\right]$

Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left[f(x_1) + 4f(x_2) + 2f(x_3) + 4f(x_4) + \ldots + 2f(x_{n-1}) + 4f(x_n) + f(x_{n+1})\right]$$

with Error $\leq \dfrac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right]$

Arc Length:

$$L = \int_a^b \sqrt{1 + f'(x)^2}\;\;dx$$

Surface of Revolution:

$$S = 2\pi \int_a^b f(x)\sqrt{1 + f'(x)^2}\;\;dx$$

(where $f(x) \geq 0$)

$$S = 2\pi \int_a^b x\sqrt{1 + f'(x)^2}\;\;dx$$

(where $a, b \geq 0$)

Work Done by a Variable Force:

$$W = \int_a^b F(x)\;\;dx$$

Force Exerted by a Fluid:

$$F = \int_a^b w\,d(y)\,\ell(y)\;\;dy$$

Taylor Series Expansion for $f(x)$:

$$p_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

Maclaurin Series Expansion for $f(x)$, where $c = 0$:

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

Summary of Tests for Series:

Test	Series	Condition(s) of Convergence	Condition(s) of Divergence	Comment
nth-Term	$\sum_{n=1}^{\infty} a_n$		$\lim_{n\to\infty} a_n \neq 0$	This test cannot be used to show convergence.
Geometric Series	$\sum_{n=0}^{\infty} r^n$	$\lvert r\rvert < 1$	$\lvert r\rvert \geq 1$	Sum $= \frac{1}{1-r}$
Telescoping Series	$\sum_{n=1}^{\infty} (b_n - b_{n+a})$	$\lim_{n\to\infty} b_n = L$		Sum $= \left(\sum_{n=1}^{a} b_n\right) - L$
p-Series	$\sum_{n=1}^{\infty} \frac{1}{(an+b)^p}$	$p > 1$	$p \leq 1$	
Integral Test	$\sum_{n=0}^{\infty} a_n$	$\int_1^{\infty} a(n)\, dn$ is convergent	$\int_1^{\infty} a(n)\, dn$ is divergent	$a_n = a(n)$ must be continuous
Direct Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $0 \leq a_n \leq b_n$	$\sum_{n=0}^{\infty} b_n$ diverges and $0 \leq b_n \leq a_n$	
Limit Comparison	$\sum_{n=0}^{\infty} a_n$	$\sum_{n=0}^{\infty} b_n$ converges and $\lim_{n\to\infty} a_n/b_n \geq 0$	$\sum_{n=0}^{\infty} b_n$ diverges and $\lim_{n\to\infty} a_n/b_n > 0$	Also diverges if $\lim_{n\to\infty} a_n/b_n = \infty$
Ratio Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$	$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} a_{n+1}/a_n = \infty$
Root Test	$\sum_{n=0}^{\infty} a_n$	$\lim_{n\to\infty} (a_n)^{1/n} < 1$	$\lim_{n\to\infty} (a_n)^{1/n} > 1$	$\{a_n\}$ must be positive Also diverges if $\lim_{n\to\infty} (a_n)^{1/n} = \infty$