MATH 2000 ASSIGNMENT 5 **SOLUTIONS**

Throughout this assignment, let A, B, C and D be any sets.

1. a) $C \cap B = \{ x \mid x \in C \text{ and } x \in B\}$.
   b) $D \cup C = \{ x \mid x \in D \text{ or } x \in C\}$.
   c) $C = D$ iff $(\forall x)\ (x \in C \Leftrightarrow x \in D)$.
   d) $D \subseteq A$ iff $(\forall x)\ (x \in D \Rightarrow x \in A)$.
   e) $\overline{C} = \{x \mid x \notin C\}$.
   f) $D - C = \{x \mid x \in D \text{ and } x \notin C\}$.
   g) $C \times B = \{(x, y) \mid x \in C \text{ and } y \in D\}$.
   h) $\bigcup_{i\in I} C_i = \{x \mid x \in C_i \text{ for some } i \in I\}$.
   i) $\bigcap_{i\in I} D_i = \{x \mid x \in D_i \text{ for all } i \in I\}$.
   j) $\boldsymbol{P}(A)$ (The power set of A) $= \{B \mid B \subseteq A\}$.
   k) S is a **partition** of A S is a partition of A if it satisfies :
      1. $\forall X \in S, \quad X \neq \emptyset$
      2. $\forall X, Y \in S$, either $X = Y$ or $X \cap Y = \emptyset$
      3. $\bigcup_{X\in S} X = A$

2. a) State one of De Morgan's laws for logic:
   $\sim (P \wedge Q)$ is equivalent to $\sim P \vee \sim Q$.
   b) State one of De Morgan's laws for sets: $\overline{A \cap B} = \overline{A} \cup \overline{B}$.
   c) State one of the Distributive Laws for logic:
   $P \wedge (Q \vee R)$ is equivalent to $(P \wedge Q) \vee (P \wedge R)$.
   d) State the tautology for the negation of an implication:
   $\sim (P \Rightarrow Q)$ is equivalent to $P \wedge \sim Q$.

3. For each i in the natural numbers, let $A_i = \{i, 2i, 3i\}$, and
   $B_i = \{-i, -i + 1, -i + 2, \ldots, 0\}$. Find the following sets:
   a) $A_1 = \{1, 2, 3\}$, $A_2 = \{2, 4, 6\}$ and $A_3 = \{3, 6, 9\}$.
   b) $B_1 = \{-1, 0\}$, $B_2 = \{-2, -1, 0\}$ and $B_3 = \{-3, -2, -1, 0\}$.
   c) $\bigcup_{i=1}^{3} A_i = \{1, 2, 3, 4, 6, 9\}$.
   d) $\bigcup_{i=1}^{\infty} A_i = \boldsymbol{N}$.
   e) $\bigcap_{i=1}^{\infty} A_i = \{ \ \} = \emptyset$.
   f) $A_1 \cap B_1 = \emptyset$.
   g) $\bigcup_{i=1}^{4} B_i = \{-4, -3, -2, -1, 0\}$.
   h) $\bigcup_{i=1}^{\infty} B_i = \{ \ldots, -2, -1, 0\}$.
   i) $\bigcap_{i=1}^{\infty} B_i = \{-1, 0\}$.

4. Page 29 Question 1.17

Let $U = \{1, 3, \ldots, 15\}$ be the universal set, $A = \{1, 5, 9, 13\}$, and $B = \{3, 9, 15\}$.
Determine the following :

a) $A \cup B = \{1, 3, 5, 9, 13, 15\}$ b) $A \cap B = \{9\}$ c) $A - B = \{1, 5, 13\}$
d) $B - A = \{3, 15\}$ e) $\overline{A} = \{3, 7, 11, 15\}$ f) $A \cap \overline{B} = \{1, 5, 13\}$

Page 30 Question 1.26
For a real number r, define $A_r = \{r^2\}$, B_r as the closed interval $[r-1,\ r+1]$, and C_r as the interval (r, ∞). For $S = \{1, 2, 4\}$, determine:

a) $\bigcup_{\alpha \in S} A_\alpha = A_1 \cup A_2 \cup A_4 = \{1\} \cup \{4\} \cup \{16\} = \{1, 4, 16\}$ and
$\bigcap_{\alpha \in S} A_\alpha = A_1 \cap A_2 \cap A_4 = \{1\} \cap \{4\} \cap \{16\} = \emptyset$

b) $\bigcup_{\alpha \in S} B_\alpha = B_1 \cup B_2 \cup B_4 = [0, 2] \cup [1, 3] \cup [3, 5] = [0, 5]$ and
$\bigcap_{\alpha \in S} B_\alpha = B_1 \cap B_2 \cap B_4 = [0, 2] \cap [1, 3] \cap [3, 5] = \emptyset$

c) $\bigcup_{\alpha \in S} C_\alpha = C_1 \cup C_2 \cup C_4 = (1, \infty) \cup (2, \infty) \cup (4, \infty) = (1, \infty)$ and
$\bigcap_{\alpha \in S} C_\alpha = C_1 \cap C_2 \cap C_4 = (1, \infty) \cap (2, \infty) \cap (4, \infty) = (4, \infty)$

Question 1.29
Let $A = \{a, b, \ldots, z\}$ be the set consisting of the letters of the alphabet. For $\alpha \in A$, let A_α consist of α and the two letters that follow it, where $A_y = \{y, z, a\}$ and $A_z = \{z, a, b\}$. Find a set $S \subseteq A$ of smallest cardinality such that $\bigcup_{\alpha \in S} A_\alpha = A$. Explain why your set S has the required properties.

Solution: Since there are 26 letters in the alphabet, $|A| = 26$.
Now since $|A_\alpha| = 3$, $|S|$ must be at least 9. So let $S = \{a, d, g, j, m, p, s, v, y\}$.
Then $\bigcup_{\alpha \in S} A_\alpha = A_a \cup A_d \cup A_g \cup A_j \cup A_m \cup A_p \cup A_s \cup A_v \cup A_y$
$= \{a, b, c\} \cup \{d, e, f\} \cup \{g, h, i\} \cup \{j, k, l\} \cup \{m, n, o\} \cup \{p, q, r\} \cup$
$\{s, t, u\} \cup \{v, w, x\} \cup \{y, z, a\}$
$= A$

Question 1.32 Which of the following are partitions of $A = \{a, b, c, d, e, f, g\}$?

a) $S_1 = \{\{a, c, e, g\}, \{b, f\}, \{d\}\}$ is a partition
b) $S_2 = \{\{a, b, c, d\}, \{e, f\}\}$ is not a partition since $g \notin \bigcup S_2$.
c) $S_3 = \{\{A\}\}$ is a partition
d) $S_4 = \{\{a\}, \emptyset, \{b, c, d\}, \{e, f, g\}\}$ is not a partition since $\emptyset \in S_4$
e) $S_5 = \{\{a, c, d\}, \{b, g\}, \{e\}, \{b, f\}\}$ is not a partition since $\{b, g\} \cap \{b, f\} \neq \emptyset$

Question 1.34 Let $A = \{1, 2, 3, 4, 5, 6\}$.
Give an example of a partition S of A such that $|S| = 3$. eg $S = \Big\{\{1, 2, 3\}, \{4\}, \{5, 6\}\Big\}$

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Question 1.36 Give an example of three sets A, S_1, and S_2 such that S_1 is a partition of A, S_2 is a partition of S_1 and $|S_2| < |S_1| < |A|$.
eg $A = \{1, 2, 3, 4\}$ $S_1 = \Big\{\{1\}, \{2\}, \{3, 4\}\Big\}$ and $S_2 = \Big\{\Big\{\{1\}, \{2\}\Big\}, \Big\{\{3, 4\}\Big\}\Big\}$

Question 1.38 Give an example of a partition of $\boldsymbol{N}$ into three subsets.
There are many possibilities. Here are a couple.
eg $S = \Big\{\{1,2,3\},\{4,5,6,7\},\{n : n \geq 8\}\Big\}$
eg $S = \Big\{\{n : n \equiv 0(\text{mod } 3)\},\{n : n \equiv 1(\text{mod } 3)\},\{n : n \equiv 2(\text{mod } 3)\}\Big\}$.

Question 1.41 Let $A = \{x,y,z\}$ and $B = \{x,y\}$. Determine $A \times B$.
$A \times B = \{(x,x),\ (x,y),\ (y,x),\ (y,y),\ (z,x),\ (z,y)\}$

Question 1.42 Let $A = \{1,\ \{1\},\ \{\{1\}\}\}$. Determine $A \times A$.

$$A \times A = \{\ (1,1),\ (1,\{1\}),\ (1,\{\{1\}\}),\ (\{1\},1),\ (\{1\},\{1\}),\ (\{1\},\{\{1\}\}),\ (\{\{1\}\},1),\ (\{\{1\}\},\{1\}),\ (\{\{1\}\},\{\{1\}\})\ \}$$

Question 1.43 Let $A = \{a,b\}$. Determine $A \times \boldsymbol{P}(A)$.
$\boldsymbol{P}(A) = \{\emptyset,\ \{a\},\ \{b\},\ A\ \}$
$A \times \boldsymbol{P}(A) = \{(a,\emptyset),\ (a,\{a\}),\ (a,\ \{b\}),\ (a,A),\ (b,\emptyset),\ (b,\{a\}),\ (b,\ \{b\}),\ (b,A)\}$

Question 1.44 Let $A = \{\emptyset,\{\emptyset\}\}$. Determine $A \times \boldsymbol{P}(A)$.
$\boldsymbol{P}(A) = \{\emptyset,\ \{\emptyset\},\ \{\{\emptyset\}\},\ A\ \}$
$A \times \boldsymbol{P}(A) = \{(\emptyset,\emptyset),\ (\emptyset,\{\emptyset\}),\ (\emptyset,\ \{\{\emptyset\}\}),\ (\emptyset,A),\ (\{\emptyset\},\emptyset),\ (\{\emptyset\},\{\emptyset\}),\ (\{\emptyset\},\ \{\{\emptyset\}\}),\ (\{\emptyset\},A)\}$

5. Page 103 Question 4.27 Let A and B be sets.
Prove that $A \cup B = (A - B) \cup (B - A) \cup (A \cap B)$.

PROOF: This is a set equality proof, for-all-iff.

Let $x \in U$.
$x \in A \cup B$
$\Leftrightarrow x \in A$ or $x \in B$
$\Leftrightarrow (x \in A$ and $(x \in B$ or $x \notin B))$ or $(x \in B$ and $(x \in A$ or $x \notin A))$
since $P \wedge$ TRUE $\Leftrightarrow P$.
$\Leftrightarrow (x \in A$ and $x \in B)$ or $(x \in A$ and $x \notin B)$ or
$(x \in B$ and $x \in A)$ or $(x \in B$ and $x \notin A)$ by distributive law
$\Leftrightarrow x \in A \cap B$ or $x \in A - B$ or $x \in A \cap B$ or $x \in B - A$
$\Leftrightarrow x \in A - B$ or $x \in B - A$ or $x \in A \cap B$ by idempotence,commutative and associative
$\Leftrightarrow x \in \Big((A - B) \cup (B - A) \cup (A \cap B)\Big)$.

Question 4.29 Let A and B be sets. Prove that $A \cap B = A$ if and only if $A \subseteq B$.

PROOF: This is an iff set proof; we break it into two parts

Part I: Prove that if $A \cap B = A$ then $A \subseteq B$.

PROOF: This is an If-Then set proof.

ASSUME: $A \cap B = A$ SAVE this for later use.
GOAL: $A \subseteq B$. Subset proof

Let $x \in U$.
$x \in A \Rightarrow x \in A \cap B$ by the assumption
$\Rightarrow x \in A$ and $x \in B$
$\Rightarrow x \in B$ by removing clauses

Part II. Prove that if $A \subseteq B$ then $A \cap B = A$.

PROOF: This is an If-Then set proof.

ASSUME: $A \subseteq B$, so for all x, if $x \in A$ then $x \in B$. SAVE this for later use.
GOAL: $A \cap B = A$. Set equality proof (two subset proofs)

Part A: Show $A \cap B \subseteq A$

Let $x \in U$.
$x \in A \cap B$
$\Rightarrow x \in A$ and $x \in B$
$\Rightarrow x \in A$ by removing clauses

Part B: Show $A \subseteq A \cap B$

Let $x \in U$.
$x \in A$
$\Rightarrow x \in A$ and $x \in A$ by idempotence
$\Rightarrow x \in A$ and $x \in B$ by assumption
$\Rightarrow x \in A \cap B$

Question 4.31 Prove that if A and B are sets such that $A \cup B \neq \emptyset$, then $A \neq \emptyset$ or $B \neq \emptyset$.

PROOF: This is an If-Then set proof. I will show a direct proof and a contrapositive proof.

Direct	Contrapositive
ASSUME: $A \cup B \neq \emptyset$	ASSUME: $A = \emptyset$ and $B = \emptyset$
GOAL: $A \neq \emptyset$ or $B \neq \emptyset$	GOAL: $A \cup B = \emptyset$
Let $x \in U$. Since $A \cup B \neq \emptyset$, there is an element in it. $x \in A \cup B \Rightarrow x \in A$ or $x \in B$ So there is an element in A or in B So either A is not empty or B is not. So $A \neq \emptyset$ or $B \neq \emptyset$.	Since $A = \emptyset$ and $B = \emptyset$; $A \cup B = \emptyset \cup \emptyset = \emptyset$.

Question 4.32 Let $A = \{n \in \boldsymbol{Z} : n \equiv 1 (\text{mod } 2)\}$ and $B = \{n \in \boldsymbol{Z} : n \equiv 3 (\text{mod } 4)\}$. Prove $B \subseteq A$.

PROOF: Subset proof

Let $n, k \in \boldsymbol{Z}$.

$n \in B \Rightarrow 4|(n-3) \Rightarrow n = 4k+3 \Rightarrow n = 4k+2+1 \Rightarrow n = 2(k+1)+1 \Rightarrow n \in A$ since $k+1 \in \boldsymbol{Z}$.

Question 4.34 Prove that $A \cap B = B \cap A$ for every two sets A and B.

PROOF: Set equality proof

Let $x \in U$.

$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B \Leftrightarrow x \in B$ and $x \in A \Leftrightarrow x \in B \cap A$.

Question 4.35

Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for every three sets A, B and C.

PROOF: Set equality proof

Let $x \in U$.

$x \in A \cap (B \cup C) \Leftrightarrow x \in A$ and $x \in (B \cup C)$

$\Leftrightarrow x \in A$ and ($x \in B$ or $x \in C$)

$\Leftrightarrow$ ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) by distributive law

$\Leftrightarrow x \in (A \cap B)$ or $x \in (A \cap C)$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$

Question 4.38 Let A, B and C be sets. Prove that $(A-B)\cup(A-C) = A-(B\cap C)$.

PROOF: Set equality proof using previous results

$(A-B)\cup(A-C) = (A\cap\overline{B})\cup(A\cap\overline{C}) = A\cap(\overline{B}\cup\overline{C}) = A\cap(\overline{B\cap C}) = A-(B\cap C)$.

Question 4.39 Let A, B and C be sets. Prove that $\overline{\overline{A}\cup(\overline{B}\cap C)} = (A\cap B)\cup(A-C)$.

PROOF: Set equality proof using previous results

$$
\begin{aligned}
\overline{\overline{A}\cup(\overline{B}\cap C)} &= \overline{\overline{A}}\cap(\overline{\overline{B}\cap C}) \\
&= A\cap(\overline{\overline{B}}\cup\overline{C}) \\
&= A\cap(B\cup\overline{C}) \\
&= (A\cap B)\cup(A\cap\overline{C}) \\
&= (A\cap B)\cup(A-C).
\end{aligned}
$$

Question 4.40 Let A and B be sets.
Prove that $A\times B = \emptyset$ if and only if $A = \emptyset$ or $B = \emptyset$.

PROOF: If and only if set proof so break into two parts.

PART I :
Prove if $A\times B=\emptyset$ then $A=\emptyset$ or $B=\emptyset$

Proof by contradiction
ASSUME: $A\times B = \emptyset$ and
$A \neq \emptyset$ and $B \neq \emptyset$
Let $x, y \in U$
$A \neq \emptyset \Rightarrow \exists x\in A$ and
$B \neq \emptyset \Rightarrow \exists y \in B$
Therefore $(x, y)\in A\times B$
This is a contradiction since $A\times B=\emptyset$
Therefore if $A\times B = \emptyset$ then
$A = \emptyset$ or $B = \emptyset$.

PART II:
Prove if $A=\emptyset$ or $B=\emptyset$ then $A\times B=\emptyset$.

ASSUME: $A = \emptyset$ or $B = \emptyset$
GOAL: $A\times B=\emptyset$.

WLOG pick $A=\emptyset$
So $A\times B=\emptyset\times B = \emptyset$.

Question 4.43 Let A, B and C be nonempty sets.
Prove that $A \times C \subseteq B \times C$ if and only if $A \subseteq B$.

PROOF: If and only if set proof so break into two parts.

PART I:
Prove if $A \times C \subseteq B \times C$, then $A \subseteq B$.

ASSUME: $A \times C \subseteq B \times C$
GOAL: $A \subseteq B$.

Let $x, y \in U$ Note $\exists y \in C$ since $C \neq \emptyset$
$x \in A \Rightarrow (x, y) \in A \times C$ for some $y \in C$
$\Rightarrow (x, y) \in B \times C$ by assumption
$\Rightarrow x \in B$ and $y \in C$.
$\Rightarrow x \in B$ by removing clauses

PART II:
Prove if $A \subseteq B$, then $A \times C \subseteq B \times C$.

ASSUME: $A \subseteq B$
GOAL: $A \times C \subseteq B \times C$.

Let $(x, y) \in U$
$(x, y) \in A \times C \Rightarrow x \in A$ and $y \in C$
$\Rightarrow x \in B$ and $y \in C$ by assumption
$\Rightarrow (x, y) \in B \times C$.

Question 4.46 Let A, B, C and D be sets.
Prove that $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$.

PROOF: Set equality proof

Let $(x, y) \in U$
$(x, y) \in (A \times B) \cap (C \times D)$
$\Leftrightarrow (x, y) \in A \times B$ and $(x, y) \in C \times D$
$\Leftrightarrow x \in A$ and $y \in B$ and $x \in C$ and $y \in D$
$\Leftrightarrow x \in A$ and $x \in C$ and $y \in B$ and $y \in D$ by commutativity and associativity
$\Leftrightarrow x \in (A \cap C)$ and $y \in (B \cap D)$
$\Leftrightarrow (x, y) \in (A \cap C) \times (B \cap D)$.

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Question 5.19 Let $S = \{p + q\sqrt{2} \ : \ p, q \in \boldsymbol{Q}\}$ and $T = \{r + s\sqrt{3} \ : \ r, s \in \boldsymbol{Q}\}$.
Prove that $S \cap T = \boldsymbol{Q}$.

PROOF: Set equality proof (two subset proofs)

PART I:
Prove $S \cap T \subseteq \boldsymbol{Q}$ by contradiction

ASSUME: $S \cap T \nsubseteq \boldsymbol{Q}$
That is $\exists x \in S \cap T$ and $x \notin \boldsymbol{Q}$

$x \in S \cap T$
$\Rightarrow x \in S$ and $x \in T$
$\Rightarrow x = p + q\sqrt{2}$ and $x = r + s\sqrt{3}$
If $q = 0$ or $s = 0$ then $x \in \boldsymbol{Q}$
which is a contradiction
If $q \neq 0$ and $s \neq 0$ then $p + q\sqrt{2} = r + s\sqrt{3}$
And $p - r = s\sqrt{3} - q\sqrt{2}$
Squaring both sides gives
$(p - r)^2 = 3s^2 + 2q^2 - 2sq\sqrt{6}$
Rearranging we get $\sqrt{6} = \frac{(p-r)^2 - 3s^2 - 2q^2}{-2sq}$
contradiction since $\sqrt{6}$ is irrational
Therefore $S \cap T \subseteq \boldsymbol{Q}$.

PART II:
Prove $\boldsymbol{Q} \subseteq S \cap T$

Let $x \in \boldsymbol{Q}$
Now $x = x + (0)(\sqrt{2})$ so $x \in S$
And $x = x + (0)(\sqrt{3})$ so $x \in T$
So $x \in S$ and $x \in T$

Therefore $x \in S \cap T$

6. Prove that $(A - B) \cup (B - A) \subseteq \overline{A \cap B}$.

PROOF: This is a Subset Proof, so Direct proof of for-all-if-then.

Let $x \in U$.

$$
\begin{aligned}
& x \in (A - B) \cup (B - A) \\
\Rightarrow\ & x \in A - B \text{ or } x \in B - A \\
\Rightarrow\ & (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A) \\
\Rightarrow\ & \Big((x \in A \text{ and } x \notin B) \text{ or } x \in B\Big) \text{ and } \Big((x \in A \text{ and } x \notin B) \text{ or } x \notin A\Big) \\
& \hfill \text{by distributive law} \\
\Rightarrow\ & \Big((x \in A \text{ or } x \in B) \text{ and } (x \notin B \text{ or } x \in B)\Big) \\
& \quad \text{and } \Big((x \in A \text{ or } x \notin A) \text{ and } (x \notin B \text{ or } x \notin A)\Big) \quad \text{by distributive law} \\
\Rightarrow\ & (x \in A \text{ or } x \in B) \text{ and (TRUE) and (TRUE) and } (x \notin B \text{ or } x \notin A) \\
\Rightarrow\ & (x \in A \text{ or } x \in B) \text{ and } (x \notin B \text{ or } x \notin A) \quad \text{since } P \wedge \text{TRUE} \Rightarrow P \\
\Rightarrow\ & x \notin B \text{ or } x \notin A \quad \text{by removing clauses} \\
\Rightarrow\ & x \notin A \text{ or } x \notin B \quad \text{by commutativity} \\
\Rightarrow\ & \sim (x \in A \text{ and } x \in B) \quad \text{by DeMorgan's} \\
\Rightarrow\ & \sim (x \in A \cap B) \\
\Rightarrow\ & x \notin A \cap B \\
\Rightarrow\ & x \in \overline{A \cap B}.
\end{aligned}
$$

7. Prove that $\emptyset - A = \emptyset$.

PROOF: Use a contradiction proof.

ASSUME: $\emptyset - A \neq \emptyset$.
Since $\emptyset - A \neq \emptyset$ there is an element in $\emptyset - A$
$x \in \emptyset - A \Rightarrow x \in \emptyset$ and $x \notin A$
But this is a contradiction since $\emptyset$ does not contain any elements.
Therefore $\emptyset - A = \emptyset$.

8. Prove that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

PROOF: This is a set equality proof, for-all-iff.

Let $x \in U$.

$$
\begin{aligned}
x \in A \cup (B \cap C) \quad \Leftrightarrow\ & x \in A \text{ OR } (x \in B \text{ and } x \in C) \\
\Leftrightarrow\ & (x \in A \text{ or } x \in B) \text{ and } (x \in A \text{ or } x \in C) \\
& \hfill \text{by distributive law} \\
\Leftrightarrow\ & x \in A \cup B \text{ and } x \in A \cup C \\
\Leftrightarrow\ & x \in (A \cup B) \cap (A \cup C).
\end{aligned}
$$

9. Prove that $(A - C) - (B - C) = (A - B) - C$.

PROOF: This is a set equality proof, for-all-iff.

Let $x \in U$.

$x \in (A - C) - (B - C)$

$\Leftrightarrow$ $(x \in A$ and $x \notin C)$ and $\sim (x \in B$ and $x \notin C)$.

$\Leftrightarrow$ $(x \in A$ and $x \notin C)$ and $(x \notin B$ or $x \in C)$ by De Morgan's law

$\Leftrightarrow$ $(x \in A$ and $x \notin C$ and $x \notin B)$ or $(x \in A$ and $x \notin C$ and $x \in C)$, by distributive law.

$\Leftrightarrow$ $x \in A$ and $x \notin C$ and $x \notin B$, by $P \vee$ FALSE equivalent to P.

$\Leftrightarrow$ $x \in A$ and $x \notin B$ and $x \notin C$, by commutativity

$\Leftrightarrow$ $(x \in A - B)$ and $x \notin C$

$\Leftrightarrow$ $x \in (A - B) - C$.

10. Prove that $(A \cap B) - (A - C) = A \cap (B \cap C)$.

PROOF: This is a set equality proof, for-all-iff.

Let $x \in U$.

$x \in (A \cap B) - (A - C)$

$\Leftrightarrow$ $(x \in A$ and $x \in B)$ and $\sim (x \in A$ and $x \notin C)$

$\Leftrightarrow$ $(x \in A$ and $x \in B)$ and $(x \notin A$ or $x \in C)$ by De Morgan's law

$\Leftrightarrow$ $(x \in A$ and $x \in B$ and $x \notin A)$ or $(x \in A$ and $x \in B$ and $x \in C)$ by distributive law.

$\Leftrightarrow$ $x \in A$ and $x \in B$ and $x \in C$ since FALSE $\vee P$ is equivalent to P.

$\Leftrightarrow$ $x \in A$ and $(x \in B$ and $x \in C)$

$\Leftrightarrow$ $x \in A \cap (B \cap C)$.

11. Prove that if $A \subseteq B$ and $C \subseteq D$ then $A - D \subseteq B - C$.

PROOF: This is a If-Then set proof

ASSUME: $A \subseteq B$, i.e., for all x, if $x \in A$ then $x \in B$.
and $C \subseteq D$, i.e., for all x, if $x \in C$ then $x \in D$.
Keep these assumptions to use later.

GOAL: show that $A - D \subseteq B - C$. This is a subset proof; for-all-if-then

Let $x \in U$.

$x \in A - D$ $\Rightarrow$ $x \in A$ and $x \notin D$

$\Rightarrow$ $x \in B$ and $x \notin C$ by 1st assumption and contrapositive of 2nd assumption.

$\Rightarrow$ $x \in B - C$.

12. Prove that if $A \subseteq B$ then $B - (B - A) = A$

PROOF: This is a If-Then set proof.

ASSUME: $A \subseteq B$, so $\forall x$, if $x \in A$ then $x \in B$.
SAVE this for later use!

GOAL: $B - (B - A) = A$.

This is a set equality proof, for-all-iff. Break into two parts

Part I : $B - (B - A) \subseteq A$.

Let $x \in U$.

$x \in B - (B - A)$

$\Rightarrow$ $x \in B$ and $x \notin (B - A)$

$\Rightarrow$ $x \in B$ and $\sim (x \in B - A)$

$\Rightarrow$ $x \in B$ and $\sim (x \in B$ and $x \notin A)$

$\Rightarrow$ $x \in B$ and $(x \notin B$ or $x \in A)$ by De Morgan's

$\Rightarrow$ $(x \in B$ and $x \notin B)$ or $(x \in B$ and $x \in A)$ by Distributive

$\Rightarrow$ $(x \in B$ and $x \in A)$, using FALSE $\vee P$ equivalent to P

$\Rightarrow$ $x \in A$ using removing clauses.

Part II. $A \subseteq B - (B - A)$.

Let $x \in U$.

$x \in A$

$\Rightarrow$ $(x \in A$ and $x \in A)$, using Idempotence

$\Rightarrow$ $(x \in B$ and $x \in A)$, by the assumption

$\Rightarrow$ $(x \in B$ and $x \notin B)$ or $(x \in B$ and $x \in A)$ using FALSE $\vee P \Leftrightarrow P$

$\Rightarrow$ $x \in B$ and $(x \notin B$ or $x \in A)$ by Distributive

$\Rightarrow$ $x \in B$ and $\sim (x \in B$ and $x \notin A)$ by De Morgan's

$\Rightarrow$ $x \in B$ and $\sim (x \in B - A)$

$\Rightarrow$ $x \in B$ and $x \notin (B - A)$

$\Rightarrow$ $x \in B - (B - A)$.

13. Prove that $A \cup B \subseteq B$ iff $A \subseteq A \cap B$.

PROOF: This is an iff set proof; we break it into two parts

Part I: Prove that if $A \cup B \subseteq B$ then $A \subseteq A \cap B$.

PROOF: This is an If-Then set proof.

ASSUME: $A \cup B \subseteq B$, so for all x, if $x \in A \cup B$ then $x \in B$.
SAVE this for later use.

GOAL: $A \subseteq A \cap B$.

Let $x \in U$.

$$\begin{aligned} x \in A &\Rightarrow x \in A \text{ and } x \in A \quad \text{by idempotence} \\ &\Rightarrow x \in A \text{ and } (x \in A \text{ or } x \in B), \quad \text{by adding clauses} \\ &\Rightarrow x \in A \text{ and } x \in A \cup B \\ &\Rightarrow x \in A \text{ and } x \in B, \quad \text{by the assumption} \\ &\Rightarrow x \in A \cap B. \end{aligned}$$

Part II. Prove that if $A \subseteq A \cap B$ then $A \cup B \subseteq B$.

PROOF: This is an If-Then set proof.

ASSUME: $A \subseteq A \cap B$, so for all x, if $x \in A$ then $x \in A \cap B$.
SAVE this for later use.

GOAL: $A \cup B \subseteq B$.

Let $x \in U$.

$$\begin{aligned} x \in A \cup B &\Rightarrow x \in A \text{ or } x \in B \\ &\Rightarrow x \in A \cap B \text{ or } x \in B, \quad \text{by assumption} \\ &\Rightarrow (x \in A \text{ and } x \in B) \text{ or } x \in B \\ &\Rightarrow x \in B \text{ or } x \in B, \quad \text{by removing clauses} \\ &\Rightarrow x \in B. \end{aligned}$$

14. Prove that $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$.

PROOF: This is a subset proof, for-all-if-then.
NOTE typical elements are ordered pairs!

Let $(x,y) \in U$.
$(x,y) \in (A \times B) \cup (C \times D)$
$\Rightarrow (x,y) \in A \times B$ or $(x,y) \in C \times D$
$\Rightarrow (x \in A$ and $y \in B)$ or $(x \in C$ and $y \in D)$
$\Rightarrow (x \in A$ or $x \in C)$ and $(y \in B$ or $x \in C)$ and
$(x \in A$ or $y \in D)$ and $(y \in B$ or $y \in D)$ by distributive law, twice
$\Rightarrow (x \in A$ or $x \in C)$ and $(y \in B$ or $y \in D)$, by removing clauses
$\Rightarrow x \in A \cup C$ and $y \in B \cup D$
$\Rightarrow (x,y) \in (A \cup C) \times (B \cup D)$.

15. Prove that if $A \subseteq B$ and $C \subseteq D$ then $A \times (B - D) \subseteq B \times (B - C)$.

PROOF: This is an if-then set proof.
ASSUME: $A \subseteq B$, so $\forall x$, if $x \in A$ then $x \in B$;
and $C \subseteq D$, so $\forall x$, if $x \in C$ then $x \in D$.
SAVE this for later use!

GOAL: $A \times (B - D) \subseteq B \times (B - C)$.
This is a subset proof, for-all-if-then statement. Typical elements are ordered pairs.

Let $(x,y) \in U$.
$(x,y) \in A \times (B - D)$
$\Rightarrow$ $x \in A$ and $y \in B - D$
$\Rightarrow$ $x \in A$ and $(y \in B$ and $y \notin D)$
$\Rightarrow$ $x \in B$ and $(y \in B$ and $y \notin D)$ by assumption $x \in A \Rightarrow x \in B$
$\Rightarrow$ $x \in B$ and $(y \in B$ and $y \notin C)$
by CONTRAPOSITIVE of the assumption $x \in C \Rightarrow x \in D$
$\Rightarrow$ $x \in B$ and $y \in B - C$
$\Rightarrow$ $(x,y) \in B \times (B - C)$.

16. Let A_i be a family of sets indexed by set I.
Prove that $B - (\bigcup_{i\in I} A_i) = \bigcap_{i\in I} (B \cap \overline{A_i})$.

PROOF: This is a set equality, so a for-all-iff.

Let $x \in U$.

$x \in B - (\bigcup_{i\in I} A_i)$
$\Leftrightarrow$ $x \in B$ and $\sim (x \in \bigcup_{i\in I} A_i)$
$\Leftrightarrow$ $x \in B$ and $\sim (\exists i \in I)\ (x \in A_i)$
$\Leftrightarrow$ $x \in B$ and $(\forall i \in I) \sim (x \in A_i)$
$\Leftrightarrow$ $x \in B$ and $(\forall i \in I)\ (x \notin A_i)$
$\Leftrightarrow$ $x \in B$ and $(\forall i \in I)\ (x \in \overline{A_i})$
$\Leftrightarrow$ $(\forall i \in I)\ (x \in B$ and $x \in \overline{A_i})$
$\Leftrightarrow$ $(\forall i \in I)\ (x \in B \cap \overline{A_i})$
$\Leftrightarrow$ $x \in \bigcap_{i\in I} (B \cap \overline{A_i})$.

17. Prove that $B \cup (\bigcap_{i\in I} A_i) = \bigcap_{i\in I} (B \cup A_i)$.

PROOF: This is a set equality, so a for-all-iff.

Let $x \in U$.

$x \in B \cup (\bigcap_{i\in I} A_i)$
$\Leftrightarrow$ $x \in B$ or $(x \in \bigcap_{i\in I} A_i)$
$\Leftrightarrow$ $x \in B$ or $(\forall i \in I)\ (x \in A_i)$
$\Leftrightarrow$ $(\forall i \in I)\ (x \in B$ or $x \in A_i)$
$\Leftrightarrow$ $x \in \bigcap_{i\in I} (B \cup A_i)$.

18. Let A_i and B_i be two families of sets, with indices $i \in I$.
Prove that $(\bigcup_{i\in I} B_i) \cup (\bigcup_{i\in I} A_i) = \bigcup_{i\in I} (B_i \cup A_i)$.

PROOF: This is a set equality, so a for-all-iff.

Let $x \in U$.

$x \in (\bigcup_{i\in I} B_i) \cup (\bigcup_{i\in I} A_i)$
$\Leftrightarrow$ $(x \in \bigcup_{i\in I} B_i)$ or $(x \in \bigcup_{i\in I} A_i)$
$\Leftrightarrow$ $(\exists i \in I)(x \in B_i)$ or $(\exists i \in I)\ (x \in A_i)$
$\Leftrightarrow$ $(\exists i \in I)\ (x \in B_i$ or $x \in A_i)$
$\Leftrightarrow$ $(\exists i \in I)\ (x \in B_i \cup A_i)$
$\Leftrightarrow$ $x \in \bigcup_{i\in I} (B_i \cup A_i)$.

19. Let C be any set.
a) Prove that if $C \subseteq A$ then $C \subseteq A \cup B$.

PROOF:
a) This is an if then set proof.

ASSUME: that $C \subseteq A$; this means $(\forall x)(x \in C \Rightarrow x \in A)$. Save for later!

GOAL: show $C \subseteq A \cup B$.
This is a subset proof, for-all-if-then proof.

Let $x \in U$.
$x \in C \Rightarrow x \in A$, by the assumption
$\Rightarrow x \in A$ or $x \in B$, by adding clauses.
$\Rightarrow x \in A \cup B$.

b) Prove that $\boldsymbol{P}(A) \subseteq \boldsymbol{P}(A \cup B)$.

PROOF:
This is a subset proof. But note that typical elements here are SETS!

Let $C \in U$.
$C \in \boldsymbol{P}(A) \Rightarrow C \subseteq A$
$\Rightarrow C \subseteq A \cup B$, by part a)
$\Rightarrow C \in \boldsymbol{P}(A \cup B)$.

20. a) Prove that if $A \subseteq B$ and $S \subseteq A \cap C$, then $S \subseteq B \cap C$.

PROOF:
This is an if then set proof.

ASSUME: that $A \subseteq B$ and $S \subseteq A \cap C$;
this means $(\forall x)(x \in A \Rightarrow x \in B)$, and $(\forall x)(x \in S \Rightarrow x \in A \cap C)$.

GOAL: show $S \subseteq B \cap C$.
This is a subset proof, a for-all-if-then.

Let $x \in U$.
$x \in S \Rightarrow x \in A \cap C$, by assumption that $S \subseteq A \cap C$,
$\Rightarrow x \in A$ and $x \in C$,
$\Rightarrow x \in B$ and $x \in C$, by assumption $A \subseteq B$
$\Rightarrow x \in B$ or $x \in C$ by tautology $P \wedge Q \Rightarrow P \vee Q$
$\Rightarrow x \in B \cup C$.

b) Prove that if $A \subseteq B$ then $\boldsymbol{P}(A \cap C) \subseteq \boldsymbol{P}(B \cap C)$.

PROOF:
This is an if-then set proof.

ASSUME: that $A \subseteq B$, save this for later.

GOAL: Show that $\boldsymbol{P}(A \cap C) \subseteq \boldsymbol{P}(B \cup C)$.
Note: typical elements here are SETS!

Let $S \in U$.
$S \in \boldsymbol{P}(A \cap C) \Rightarrow S \subseteq A \cap C$
$\Rightarrow S \subseteq B \cup C$, by assumption $A \subseteq B$ and part a)
$\Rightarrow S \in \boldsymbol{P}(B \cup C)$.