1. Let $f : A \to B$ be a function. Write definitions for the following **in logical form**, with negations worked through.

   (a) f is one-to-one iff $\forall x, y \in A$, if $f(x) = f(y)$ then $x = y$.

   (b) f is onto B iff $\forall w \in B$, $\exists x \in A$ such that $f(x) = w$.

   (c) f is not one-to-one iff $\exists x, y \in A$ such that $f(x) = f(y)$ but $x \neq y$.

   (d) f is not onto B iff $\exists w \in B$ such that $\forall x \in A$, $f(x) \neq w$.

2. For each of the following, give an example of sets A, B and C and functions $f : A \to B$ and $g : B \to C$ which satisfy the given conditions. NOTE: you do not need to give formulas on $\boldsymbol{Z}$ or $\boldsymbol{R}$; it is much easier to draw pictures of small sets and indicate your functions on the pictures.

   (a) f is one-to-one but not onto, and g is onto but not one-to-one.
   Example: Let $A = \{a, b\}$, $B = \{p, q, r\}$ and $C = \{x, y\}$,
   with $f = \{(a, p), (b, q)\}$ and $g = \{(p, x), (q, y), (r, y)\}$.

   (b) g is onto C, but $g \circ f$ is not onto C.
   Example: Let $A = \{a\}$, $B = \{p, q\}$ and $C = \{x, y\}$,
   with $f = \{(a, p)\}$ and $g = \{(p, x), (q, y)\}$.

   (c) f is not onto B, but $g \circ f$ is onto C.
   Example: Let $A = \{a, b\}$, $B = \{p, q, r\}$ and $C = \{x, y\}$,
   with $f = \{(a, p), (b, q)\}$ and $g = \{(p, x), (q, y), (r, y)\}$.

   (d) f is one-to-one, but $g \circ f$ is not one-to-one.
   Example: Let $A = \{a, b\}$, $B = \{p, q, r\}$ and $C = \{x\}$,
   with $f = \{(a, p), (b, q)\}$ and $g = \{(p, x), (q, x), (r, x)\}$.

3. (a) Prove that the function $f : \boldsymbol{R} \to \boldsymbol{R}$ by $f(x) = 5x + 11$ is one-to-one.
   Proof: Let $a, b \in \boldsymbol{R}$. Then

$$\begin{aligned} f(a) = f(b) \Rightarrow \quad 5a + 11 &= 5b + 11 \\ \Rightarrow \quad 5a &= 5b \\ \Rightarrow \quad a &= b. \end{aligned}$$

   Therefore f is one-to-one.

   (b) Prove that the function $f : \boldsymbol{R} \to \boldsymbol{R}$ by $f(x) = -x^4 + 12$ is not one-to-one.
   Proof: Note that $f(1) = 11$ and $f(-1) = 11$ but $1 \neq -1$. Therefore f is not one-to-one. (Any example using a number and its negative will work here.)

(c) Prove that the function $f: \boldsymbol{R} \to \boldsymbol{R}$ given by $f(x) = 63x - 51$ maps onto its codomain $\boldsymbol{R}$.
Given any $y \in \boldsymbol{R}$, can we find an $x \in \boldsymbol{R}$ such that $f(x) = y$?
(on scrap paper) We need to have $f(x) = 63x - 51 = y$, so $x = \dfrac{y+51}{63}$.
Pick $x = \dfrac{y+51}{63}$. Note that $x \in \boldsymbol{R}$.
So $f(x) = f\left(\dfrac{y+51}{63}\right) = 63\left(\dfrac{y+51}{63}\right) - 51 = y + 51 - 51 = y.$
Therefore f maps onto its codomain $\boldsymbol{R}$.

(d) Prove that the function $f: \boldsymbol{Z} \to \boldsymbol{Z}$ given by $f(x) = 63x - 51$ does not map onto its codomain.
Given any $y \in \boldsymbol{Z}$, can we find an $x \in \boldsymbol{Z}$ such that $f(x) = y$?
Note that $x = \frac{y+51}{63}$ will not work since it is not necessarily an integer.
COUNTEREXAMPLE: Pick $y = 1$. The only x that will generate 1, is $\frac{52}{63}$ which is not an integer. So we can not get to 1. Therefore f is not onto $\boldsymbol{Z}$. (Any y that can not be generated from an integer will work here).

4. Let $f: \boldsymbol{R} \to \boldsymbol{R}$ by $f(x) = 2x^2 + 1$ and $g: \boldsymbol{R} \to \boldsymbol{R}$ by $g(x) = 3x - 10$.
Find $g \circ f$ and $f \circ g$.

$$\begin{aligned} f \circ g(x) &= f(g(x)) = f(3x-10) &= 2(3x-10)^2 + 1. \\ g \circ f(x) &= g(f(x)) = g(2x^2+1) &= 3(2x^2+1) - 10. \end{aligned}$$

5. Prove whether each of the following functions is one-to-one or not and whether it is onto its codomain or not.

(a) $f: \boldsymbol{R} \to \boldsymbol{R}$ by $f(x) = 12x^3 + 5$.

- ONE-TO-ONE: Let $a, b \in \boldsymbol{R}$. Then

$$\begin{aligned} f(a) = f(b) \Rightarrow\ 12a^3 + 5 &= 12b^3 + 5 \\ \Rightarrow\ 12a^3 &= 12b^3 \\ \Rightarrow\ a^3 &= b^3 \\ \Rightarrow\ a &= b. \end{aligned}$$

  Therefore f is one-to-one.

- ONTO: Given any $y \in \boldsymbol{R}$, can we find an $x \in \boldsymbol{R}$ such that $f(x) = y$?
  (on scrap paper) We need to have $f(x) = 12x^3 + 5 = y$, so $x = \sqrt[3]{\dfrac{y-5}{12}}$.
  Pick $x = \sqrt[3]{\dfrac{y-5}{12}}$. Note that $x \in \boldsymbol{R}$.
  So $f(x) = f\left(\sqrt[3]{\dfrac{y-5}{12}}\right) = 12\left(\sqrt[3]{\dfrac{y-5}{12}}\right)^3 + 5 = 12\left(\dfrac{y-5}{12}\right) + 5 = y.$
  Therefore f maps onto $\boldsymbol{R}$.

(b) $f : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ by $f(x, y) = 2y - 3x$.

- ONE-TO-ONE: COUNTEREXAMPLE: lots of different pairs give the same output, for instance $f(0, 1) = 2 = f(2, 4)$.
  Therefore f is NOT one-to-one.
- ONTO: Given any number $w \in \boldsymbol{R}$, can we find an input pair $(x, y) \in \boldsymbol{R} \times \boldsymbol{R}$ such that $f(x, y) = w$?
  (on scrap paper) In order to satisfy this, we would need $2y - 3x = w$. There are lots of ways to find such a pair. For instance, given w, take $x = 0$, and $y = w/2$, so use the pair $(0, w/2)$.
  Pick $(x, y) = (0, w/2)$. Note that $(0, w/2) \in \boldsymbol{R} \times \boldsymbol{R}$.
  $f(x, y) = f(0, w/2) = 2(w/2) - 3(0) = w$.
  Therefore f maps onto $\boldsymbol{R}$.

(c) $f : \boldsymbol{Z} \to \boldsymbol{Z} \times \boldsymbol{Z}$ by $f(x) = (-4x, x + 4)$.

- ONE-TO-ONE: Let $a, b \in \boldsymbol{Z}$. Then

$$\begin{aligned} f(a) = f(b) &\Rightarrow (-4a, a + 4) = (-4b, b + 4) \\ &\Rightarrow -4a = -4b \text{ and } a + 4 = b + 4 \\ &\Rightarrow a = b \text{ and } a = b \\ &\Rightarrow a = b \end{aligned}$$

  Therefore f is one-to-one.
- ONTO: COUNTEREXAMPLE: Note that in all images of this function, the first coordinates are multiples of 4; so it won't be possible to produce pairs such as (x, y) where x is not divisible by 4 eg $(3, 4)$. Therefore this function does not map onto $\boldsymbol{Z} \times \boldsymbol{Z}$.

(d) $f : \boldsymbol{Z} \to \boldsymbol{Z} \times \boldsymbol{Z}$ by $f(x) = (2x^2, x)$.

- ONE-TO-ONE: Let $a, b \in \boldsymbol{Z}$. Then

$$\begin{aligned} f(a) = f(b) &\Rightarrow (2a^2, a) = (2b^2, b) \\ &\Rightarrow 2a^2 = 2b^2 \text{ and } a = b \\ &\Rightarrow a = b \end{aligned}$$

  Therefore f is one-to-one.
- ONTO: COUNTEREXAMPLE: Note that in all images of this function the first coordinates are even; so it won't be possible to produce pairs such as (x, y) where x is odd, eg $(3, 6)$. Therefore this function does not map onto $\boldsymbol{Z} \times \boldsymbol{Z}$.

(e) $f : \boldsymbol{Z} \times \boldsymbol{Z} \to \boldsymbol{Z}$ by $f(a, b) = 3ab$.

- ONE-TO-ONE: COUNTEREXAMPLE: Note that $f(1, 4) = 12$ and $f(2, 2) = 12$ but $(1, 4) \neq (2, 2)$.
  Therefore this function is not one-to-one.
- ONTO: COUNTEREXAMPLE: Note that all images of this function are multiples of 3; so it won't be possible to produce 1 or 2. Therefore this function does not map onto $\boldsymbol{Z}$.

(f) $f : \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ by $f(x, y) = 3y + 2$.

- ONE-TO-ONE: COUNTEREXAMPLE: It is easy to find distinct pairs that give the same output. For instance $f(2, 0) = 2$ and $f(5, 0) = 2$ but $(2, 0) \neq (5, 0)$. Therefore f is not one-to-one.
- ONTO: Given any number $w \in \boldsymbol{R}$, can we find a pair $(x, y) \in \boldsymbol{R} \times \boldsymbol{R}$ such that $f(x, y) = w$?
  (on scrap paper) We need $3y + 2 = w$. Solving for y in terms of w, we get $y = \frac{w-2}{3}$. Note there is no restriction on x, so we can use any real number for x!
  Let $(x, y) = (0, \frac{w-2}{3})$. This pair is in $\boldsymbol{R} \times \boldsymbol{R}$.
  Now $f(x, y) = f(0, \frac{w-2}{3}) = 3(\frac{w-2}{3}) + 2 = (w - 2) + 2 = w$.
  Therefore f is onto $\boldsymbol{R}$.

(g) $f : \boldsymbol{Z} \to \boldsymbol{Z} \times \boldsymbol{Z}$ by $f(x) = (x + 4, x - 1)$.

- ONE-TO-ONE: Let $a, b \in \boldsymbol{Z}$. Then

$$\begin{aligned} f(a) = f(b) &\Rightarrow (a + 4, a - 1) = (b + 4, b - 1) \\ &\Rightarrow a + 4 = b + 4 \quad \text{and} \quad a - 1 = b - 1 \\ &\Rightarrow a = b \quad \text{and} \quad a = b \\ &\Rightarrow a = b. \end{aligned}$$

  Therefore f is one-to-one.
- ONTO: COUNTEREXAMPLE : There is no way to get to $(3, 3)$ since that would require that $x + 4 = 3$ and $x - 1 = 3$ which would mean that $x = -1$ and $x = 4$ at the same time which is impossible. Therefore f does not map onto $\boldsymbol{Z} \times \boldsymbol{Z}$.

(h) $f : \boldsymbol{Z} \times \boldsymbol{Z} \to \boldsymbol{Z} \times \boldsymbol{Z}$ by $f(x, y) = (-y, x - y)$.

- ONE-TO-ONE: Let $a, b, c, d \in \boldsymbol{Z}$. Then

$$\begin{aligned} f(a,b) = f(c,d) &\Rightarrow (-b, a-b) = (-d, c-d) \\ &\Rightarrow -b = -d \quad \text{and} \quad a-b = c-d \\ &\Rightarrow b = d \quad \text{and} \quad a-b = c-d \\ &\Rightarrow b = d \quad \text{and} \quad a-b = c-b \\ &\Rightarrow b = d \quad \text{and} \quad a = c \\ &\Rightarrow a = c \quad \text{and} \quad b = d \\ &\Rightarrow (a,b) = (c,d). \end{aligned}$$

  Therefore f is one-to-one.

- ONTO: Given any pair $(a, b) \in \boldsymbol{Z} \times \boldsymbol{Z}$, can we find a pair $(x, y) \in \boldsymbol{Z} \times \boldsymbol{Z}$ such that $f(x, y) = (a, b)$?
  (on scrap paper) We need $(a, b) = (-y, x-y)$. So $a = -y$ and $b = x-y$.
  So use $y = -a$ and $x = b + y = b - a$.
  Let $(x, y) = (b - a, -a)$. Note that $(b - a, -a) \in \boldsymbol{Z} \times \boldsymbol{Z}$
  Now $f(x, y) = f(b - a, -a) = \big(-(-a), (b - a) - (-a)\big) = (a, b)$.
  Therefore f is onto $\boldsymbol{Z} \times \boldsymbol{Z}$.

6. Let $f : A \to B$ and $g : B \to C$. Prove that if f is one-to-one and g is one-to-one, then $g \circ f$ is one-to-one.

   PROOF:
   ASSUME: f is one-to-one, i.e., $(\forall a, b \in A)\ \ f(a) = f(b) \Rightarrow a = b$,
   and g is one-to-one, i.e., $(\forall p, q \in B)\ \ g(p) = g(q) \Rightarrow p = q$.
   Save for later.
   Show that $g \circ f$ is one-to-one.
   Let $a, b \in A$.

$$\begin{aligned} (g \circ f)(a) = (g \circ f)(b) &\Rightarrow g(f(a)) = g(f(b)) \\ &\Rightarrow f(a) = f(b), \quad \text{since } g \text{ is one-to-one} \\ &\Rightarrow a = b, \quad \text{since } f \text{ is one-to-one} \end{aligned}$$

   Therefore $g \circ f$ is one-to-one.

7. Let $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is one-to-one then f must be one-to-one.

PROOF:
ASSUME: $g \circ f$ is one-to-one, i.e., $(\forall a, b \in A)\ \ g \circ f(a) = g \circ f(b) \Rightarrow a = b$.
Show that f is one-to-one.
Let $a, b \in A$.

$$\begin{aligned} f(a) = f(b) \Rightarrow &\quad g(f(a)) = g(f(b)), \qquad \text{since } g \text{ is a function} \\ \Rightarrow &\quad g \circ f(a) = g \circ f(b), \quad \text{by definition of } g \circ f \\ \Rightarrow &\quad a = b, \qquad \text{since } g \circ f \text{ is one-to-one.} \end{aligned}$$

Therefore f is one-to-one.

8. Let $f : A \to B$ and $g : B \to C$. Prove that if f is onto B and g is onto C, then $g \circ f$ is onto C.

PROOF:
ASSUME: f is onto B i.e., $(\forall b \in B)\ (\exists x \in A)\ \ f(x) = b]$,
and g is onto C, i.e., $(\forall w \in C)\ (\exists p \in B)\ \ g(p) = w)$.
Show that $g \circ f$ is onto C.
Let $w \in C$. Can we find an element $a \in A$ such that $g \circ f(a) = w$?
Since $w \in C$ and g maps onto C, $\exists p \in B$ such that $g(p) = w$.
Now we have $p \in B$, and since f maps onto $B, \exists a \in A$ such that $f(a) = p$.
So we have an element $a \in A$.
Now $g \circ f(a) = g(f(a)) = g(p) = w$.
Therefore $g \circ f$ is onto C

9. Let $f : A \to B$ and $g : B \to C$. Prove that if $g \circ f$ is onto C then g must be onto C.

a) PROOF:
ASSUME: $g \circ f$ is onto B i.e., $(\forall w \in C)\ (\exists x \in A)[(g \circ f)(x) = w]$.
Show that g is onto C.
Let $w \in C$. Can we find an element $b \in B$ such that $g(b) = w$?
Since $w \in C$ and $g \circ f$ maps onto C, there exists an element $a \in A$ such that $(g \circ f)(a) = w$.
Since $a \in A$ and f is a function, $\exists b \in B$ such that $f(a) = b$.
So use this element, b. $b \in B$ and $g(b) = g(f(a)) = w$.
So g is onto C.

10. Let $f : A \to B$. Define a new function $g : A \times C \to B \times C$, by the rule that $g(a, c) = (f(a), c)$, for any pair $(a, c) \in A \times C$.
Prove that if f is onto B, then g maps onto $B \times C$.

ASSUME: that f maps onto B. Save for later.
GOAL: that g maps onto $B \times C$.

Given any pair $(w, c) \in B \times C$, can we find an $(a, b) \in A \times C$ so that $g(a, b) = (w, c)$?
Let $(w, c) \in B \times C$.
Since f is onto B, $\exists a \in A$ such that $f(a) = w$. Also $c \in C$.
So use this pair, (a, c). This pair is in $A \times C$.
Now $g(a, c) = (f(a), c) = (w, c)$.
Therefore g is onto $B \times C$.

11. From the text. Pages 214-216.

- 9.14 A function $f : \boldsymbol{Z} \to \boldsymbol{Z}$ is defined by $f(n) = n - 3$.
Determine whether f is

  (a) injective : Let $a, b \in \boldsymbol{Z}$.
  Then $f(a) = f(b) \Rightarrow a - 3 = b - 3 \Rightarrow a = b$.
  Therefore f is injective.

  (b) surjective : Given any $w \in \boldsymbol{Z}$, can we find $x \in \boldsymbol{Z}$ such that $f(x) = w$?
  Pick $x = w + 3$. Then $x \in \boldsymbol{Z}$ and $f(x) = f(w + 3) = (w + 3) - 3 = w$.
  Therefore f is surjective.

- 9.17 Determine whether the function $f : \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = x^2 + 4x + 9$ is

  (a) one-to-one. f is not one-to-one since $f(-4) = 9 = f(0)$ but $-4 \neq 0$.

  (b) onto $\boldsymbol{R}$. f is not onto $\boldsymbol{R}$ since the discriminant is negative, so there are no real roots and therefore there is no x such that $f(x) = 0$.
  ALTERNATE: completing the square we get $x^2+4x+9 = (x+2)^2+5$ so $f(x)$ is always greater than or equal to 5.

- 9.18 Is there a function $f : \boldsymbol{R} \to \boldsymbol{R}$ that is onto $\boldsymbol{R}$ but not one-to-one? Explain your answer.
There are many such functions. Any continuous function with a **bump** will do.

  – eg $f(x) = x^3 - x^2$. So f is not one-to-one since $f(0) = 0 = f(1)$ but f is onto $\boldsymbol{R}$.

  – eg Define $f(x) = \begin{cases} x & x < 0 \\ -x & 0 \leq x \leq 1 \\ x - 2 & x > 1 \end{cases}$.
  Then f is not one-to-one since $f(0) = 0 = f(2)$ but f is onto $\boldsymbol{R}$.

- 9.19 Give an example of a function $f : \boldsymbol{N} \to \boldsymbol{N}$ that is
  (a) one-to-one and onto $\boldsymbol{N}$. Define $f(n) = n$
  (b) one-to-one but not onto $\boldsymbol{N}$. Define $f(n) = n + 2$.
  Then f is not onto $\boldsymbol{N}$ since there is no $n \in \boldsymbol{N}$ such that $f(n) = 1$.
  (c) onto $\boldsymbol{N}$ but not one-to-one. Define $f(n) = \begin{cases} 2 & n = 1 \\ 1 & n = 2 \\ n-2 & n > 2 \end{cases}$.
  Then f is not one-to-one since $f(1) = 2 = f(4)$.
  (d) neither one-to-one not onto $\boldsymbol{N}$. Define $f(n) = 5$. Then f is not one-to-one since $f(1) = 5 = f(2)$ and f is not onto $\boldsymbol{N}$ since we can never get any value other than 5.

- 9.20 Prove that the function $f : \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = 7x - 2$ is bijective.
  - injective : Let $a, b \in \boldsymbol{R}$.
    Then $f(a) = f(b) \Rightarrow 7a - 2 = 7b - 2 \Rightarrow a = b$.
    Therefore f is injective.
  - surjective : Given any $w \in \boldsymbol{R}$, can we find $x \in \boldsymbol{R}$ such that $f(x) = w$?
    Pick $x = \frac{w+2}{7}$. Then $x \in \boldsymbol{R}$ and $f(x) = f(\frac{w+2}{7}) = 7(\frac{w+2}{7}) - 2 = w$.
    Therefore f is surjective.

  Therefore f is a bijection.

- 9.25 Let A be a nonempty set and let $f : A \to A$ be a function.
  Prove that if $f \circ f = i_A$, then f is bijective.
  ASSUME: $f \circ f = i_A$.
  SHOW that f is bijective.
  - injective : Let $a, b \in A$. Then

$$\begin{aligned} f(a) = f(b) &\Rightarrow f(f(a)) = f(f(b)) \quad \text{since } f \text{ is a function} \\ &\Rightarrow (f \circ f)(a) = (f \circ f)(b) \\ &\Rightarrow i_A(a) = i_A(b) \quad \text{since } f \circ f = i_A \\ &\Rightarrow a = b \end{aligned}$$

    Therefore f is injective.
  - surjective : Given any $p \in A$, can we find $x \in A$ such that $f(x) = p$?
    Since $p \in A$, and f is a function, $\exists w \in A$ such that $f(p) = w$. Pick $x = w$. Then $x \in A$ and $f(x) = f(w) = f(f(p)) = (f \circ f)(p) = i_A(p) = p$. Therefore f is surjective.

  Therefore f is a bijection.

- 9.26 Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c\}$, and $C = \{w, x, y, z\}$.
  Consider the functions $f : A \to B$ and $g : B \to C$,
  where $f = \{(1, b), (2, c), (3, c), (4, a)\}$ and $g = \{(a, x), (b, y), (c, x)\}$.
  Determine $g \circ f$. $g \circ f = \{(1, y), (2, x), (3, x), (4, x)\}$

- 9.27 Two functions $f : \boldsymbol{R} \to \boldsymbol{R}$ and $g : \boldsymbol{R} \to \boldsymbol{R}$ are defined by $f(x) = 3x^2 + 1$ and $g(x) = 5x - 3$ for all $x \in \boldsymbol{R}$.
  Determine $(g \circ f)(1)$ and $(f \circ g)(1)$.
  $(g \circ f)(1) = g(f(1)) = g(4) = 17$ and $(f \circ g)(1) = f(g(1)) = f(2) = 13$

- 9.30 Let A and B be nonempty sets.
  Prove that if $f : A \to B$, then $f \circ i_A = f$ and $i_B \circ f = f$.
  $Dom(f \circ i_A) = Dom(i_A) = A = Dom(f)$. $(f \circ i_A)(x) = f(i_A(x)) = f(x)$
  $Dom(i_B \circ f) = Dom(f)$. $(i_B \circ f)(x) = i_B(f(x)) = f(x)$

- 9.33 Show that the function $f : \boldsymbol{R} \to \boldsymbol{R}$ defined by $f(x) = 4x - 3$ is bijective and determine $f^{-1}(x)$ for $x \in \boldsymbol{R}$.
  - injective : Let $a, b \in \boldsymbol{R}$.
    Then $f(a) = f(b) \Rightarrow 4a - 3 = 4b - 3 \Rightarrow a = b$. So f is injective.
  - surjective : Given any $w \in \boldsymbol{R}$, can we find $x \in \boldsymbol{R}$ such that $f(x) = w$?
    Pick $x = \frac{w+3}{4}$. Then $x \in \boldsymbol{R}$ and $f(x) = f(\frac{w+3}{4}) = 4(\frac{w+3}{4}) - 3 = w$.
    Therefore f is surjective.

  Therefore f is a bijection and $\displaystyle f^{-1}(x) = \frac{x+3}{4}$

- 9.35 Let the functions $f : \boldsymbol{R} \to \boldsymbol{R}$ and $g : \boldsymbol{R} \to \boldsymbol{R}$ be defined by $f(x) = 2x + 3$ and $g(x) = -3x + 5$.
  (a) Show that f is one-to-one and onto $\boldsymbol{R}$.
  - injective : Let $a, b \in \boldsymbol{R}$.
    Then $f(a) = f(b) \Rightarrow 2a + 3 = 2b + 3 \Rightarrow a = b$. So f is injective.
  - surjective : Given any $w \in \boldsymbol{R}$, can we find $x \in \boldsymbol{R}$ st $f(x) = w$?
    Pick $x = \frac{w-3}{2}$. Then $x \in \boldsymbol{R}$ and $f(x) = f(\frac{w-3}{2}) = 2(\frac{w-3}{2}) + 3 = w$.
    Therefore f is surjective.

  (b) Show that g is one-to-one and onto $\boldsymbol{R}$.
  - injective : Let $a, b \in \boldsymbol{R}$.
    Then $g(a) = g(b) \Rightarrow -3a + 5 = -3b + 5 \Rightarrow a = b$. So g is injective.
  - surjective : Given any $w \in \boldsymbol{R}$, can we find $x \in \boldsymbol{R}$ st $g(x) = w$?
    Pick $x = \frac{w-5}{-3}$. Then $x \in \boldsymbol{R}$; $g(x) = g(\frac{w-5}{-3}) = -3(\frac{w-5}{-3}) + 5 = w$.
    Therefore g is surjective.

  (c) Determine the composition $g \circ f$.
  $g \circ f = g(f(x)) = g(2x + 3) = -3(2x + 3) + 5 = -6x - 4$.

  (d) Determine the inverse functions f^{-1} and g^{-1}.
  $f^{-1}(x) = \frac{x-3}{2}$ and $g^{-1}(x) = \frac{x-5}{-3}$.

  (e) Determine the inverse function $(g \circ f)^{-1}$ and the compostion $f^{-1} \circ g^{-1}$.
  $(g \circ f)^{-1} = \frac{x+4}{-6}$
  $(f^{-1} \circ g^{-1})(x) = f^{-1}(g^{-1}(x)) = f^{-1}(\frac{x-5}{-3}) = \frac{\frac{x-5}{-3} - 3}{2} = \frac{(x-5)+9}{-6} = \frac{x+4}{-6}$

  Note that $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$