Math 1410–Solutions for the Midterm Practice Sheet

The Midterm Exam will be 12:00noon–2:00pm, Saturday, October 29, in PE250

1. Solve each system using Gaussian or Gauss Jordan elimination.

(a) $$\begin{cases} x^2 + y^3 - z^4 = 0 \\ x^2 + 2y^3 - 2z^4 = -9 \\ -x^2 - y^3 - z^4 = -2 \end{cases}$$

Solution:

We form the augmented matrix and find its reduced echelon form:

$$\left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 1 & 2 & -2 & -9 \\ -1 & -1 & -1 & -2 \end{array}\right]$$

$$\underset{\substack{-\text{R1}+\text{R2} \\ \text{R1}+\text{R3}}}{\sim} \left[\begin{array}{rrr|r} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$$\underset{-\text{R2}+\text{R1}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 9 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & -2 & -2 \end{array}\right]$$

$$\underset{-\frac{1}{2}\text{R3}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 9 \\ 0 & 1 & -1 & -9 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\underset{\text{R3}+\text{R2}}{\sim} \left[\begin{array}{rrr|r} 1 & 0 & 0 & 9 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Therefore, $x^2 = 9$, $y^3 = -8$, and $z^4 = 1$, which means that $x = \pm 3$, $y = -2$, and $z = \pm 1$. Writing each solution in the form (x, y, z), the four solutions for this system are

$$(3, -2, 1),\ (-3, -2, 1),\ (3, -2, -1),\ \text{and } (-3, -2, -1).$$

(b) $\begin{cases} x - 2y = b \\ -2x + 4y = -6 \end{cases}$

Solution:

As before, we form the augmented matrix and find its reduced echelon form:

$$\left[\begin{array}{rr|r} 1 & -2 & b \\ -2 & 4 & -6 \end{array}\right]$$

$$\underset{2R1+R2}{\sim} \left[\begin{array}{rr|c} 1 & -2 & b \\ 0 & 0 & 2b-6 \end{array}\right]$$

$$= \left[\begin{array}{rr|c} 1 & -2 & b \\ 0 & 0 & 2(b-3) \end{array}\right].$$

At this point, there are two possible cases.

Case 1: $b \neq 3$:

The system is inconsistent i.e. it has no solution.

Case 2: $b = 3$:

The augmented matrix becomes

$$\left[\begin{array}{rr|r} 1 & -2 & 3 \\ 0 & 0 & 0 \end{array}\right].$$

Then, y is arbitrary because there is no leading 1 in its coefficient column. Solving the first equation for x, we get $x = 3 + 2y$. Thus, the system has infinitely many solutions of the form

$$\begin{cases} x = 3 + 2y \\ y = y. \end{cases}$$

2. Solve the following matrix equation for A:

$$2A^{-1} - \begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix}^t \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}^{-1}$$

Solution:

$$2A^{-1} - \begin{bmatrix} 1 & 0 \\ 3 & 3 \end{bmatrix}^t \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^t = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}^{-1}$$

$$\Longrightarrow \quad 2A^{-1} - \begin{bmatrix} 1 & 3 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \frac{1}{(3)(2)-(-5)(-1)} \begin{bmatrix} 2 & -(-5) \\ -(-1) & 3 \end{bmatrix}$$

$$\Longrightarrow \quad 2A^{-1} - \begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

$$\Longrightarrow \quad 2A^{-1} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 3 & 3 \end{bmatrix}$$

$$\Longrightarrow \quad 2A^{-1} = \begin{bmatrix} 6 & 8 \\ 4 & 6 \end{bmatrix}$$

$$\Longrightarrow \quad \tfrac{1}{2}\left(2A^{-1}\right) = \tfrac{1}{2}\left(\begin{bmatrix} 6 & 8 \\ 4 & 6 \end{bmatrix}\right)$$

$$\Longrightarrow \quad \left(\tfrac{1}{2}(2)\right) A^{-1} = \begin{bmatrix} \tfrac{1}{2}(6) & \tfrac{1}{2}(8) \\ \tfrac{1}{2}(4) & \tfrac{1}{2}(6) \end{bmatrix}$$

$$\Longrightarrow \quad A^{-1} = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$$

$$\Longrightarrow \quad \left(A^{-1}\right)^{-1} = \frac{1}{(3)(3)-(4)(2)} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$\Longrightarrow \quad \left(A^{-1}\right)^{-1} = \frac{1}{1} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$\Longrightarrow \quad A = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}.$$

3. Find elementary matrices $E_1, E_2, \dots E_n$ such that $E_n E_{n-1} \dots E_2 E_1 A$ is the reduced echelon form of A. Also, find the products $E_n E_{n-1} \dots E_2 E_1 A$ and $E_n E_{n-1} \dots E_2 E_1$.

$$A = \begin{bmatrix} 0 & -2 & 6 & -8 \\ -1 & 1 & -1 & 3 \end{bmatrix}$$

Solution:

$$\begin{array}{rl|l}
 & \left[\begin{array}{c|c} A & I_2 \end{array}\right] & \\
= & \left[\begin{array}{rrrr|rr} 0 & -2 & 6 & -8 & 1 & 0 \\ -1 & 1 & -1 & 3 & 0 & 1 \end{array}\right] & \\
\underset{R1 \longleftrightarrow R2}{\sim} & \left[\begin{array}{rrrr|rr} -1 & 1 & -1 & 3 & 0 & 1 \\ 0 & -2 & 6 & -8 & 1 & 0 \end{array}\right] & E_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
\underset{-R1}{\sim} & \left[\begin{array}{rrrr|rr} 1 & -1 & 1 & -3 & 0 & -1 \\ 0 & -2 & 6 & -8 & 1 & 0 \end{array}\right] & E_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \\
\underset{-\frac{1}{2}R2}{\sim} & \left[\begin{array}{rrrr|rr} 1 & -1 & 1 & -3 & 0 & -1 \\ 0 & 1 & -3 & 4 & -\frac{1}{2} & 0 \end{array}\right] & E_3 = \begin{pmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \\
\underset{R2+R1}{\sim} & \left[\begin{array}{rrrr|rr} 1 & 0 & -2 & 1 & -\frac{1}{2} & -1 \\ 0 & 1 & -3 & 4 & -\frac{1}{2} & 0 \end{array}\right]. & E_4 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
\end{array}$$

Therefore, $E_4E_3E_2E_1A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -3 & 4 \end{bmatrix}$ and $E_4E_3E_2E_1 = \begin{bmatrix} -\frac{1}{2} & -1 \\ -\frac{1}{2} & 0 \end{bmatrix}$.

4. If $A = \begin{bmatrix} 0 & 0 & -2 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$, show that $A^3 + 8I = 0$. Use this to find A^{-1}.

Solution:

$$A^2 = AA = \begin{bmatrix} 0 & 0 & -2 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 4 \\ 4 & 0 & 0 \end{bmatrix}.$$

Thus, $A^3 + 8I$

$$= \left(A^2\right) A + 8I$$

$$= \begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 4 \\ 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix} + 8 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 0 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & -8 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{as required.}$$

Next, $A^3 + 8I = 0$

$$\Longrightarrow A^3 = -8I$$

$$\Longrightarrow -\tfrac{1}{8}A^3 = I$$

$$\Longrightarrow A\left(-\tfrac{1}{8}A^2\right) = I$$

$$\Longrightarrow A^{-1} = -\tfrac{1}{8}A^2$$

$$\Longrightarrow A^{-1} = -\tfrac{1}{8}\begin{bmatrix} 0 & 4 & 0 \\ 0 & 0 & 4 \\ 4 & 0 & 0 \end{bmatrix}$$

$$\Longrightarrow A^{-1} = \begin{bmatrix} 0 & -\tfrac{1}{2} & 0 \\ 0 & 0 & -\tfrac{1}{2} \\ -\tfrac{1}{2} & 0 & 0 \end{bmatrix}.$$

5. Let $A = \begin{bmatrix} 1 & 5 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 3 & 1 & 4 \end{bmatrix}$.

(a) Find the cofactors $A_{1\,3}$ and $A_{4\,3}$.

Solution:

$$A_{1\,3} = +\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 3 & 4 \end{vmatrix} = +(1)\begin{vmatrix} 0 & -2 \\ 3 & 4 \end{vmatrix} = 0-(-6) = 6, \quad \text{and}$$

$$A_{4\,3} = -\begin{vmatrix} 1 & 5 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{vmatrix} = -\left(+(-2)\begin{vmatrix} 1 & 5 \\ 1 & 0 \end{vmatrix}\right) = 2(0-5) = -10.$$

(b) Use the results above to find $|A|$.

Solution:

We use the cofactor expansion down the third column to get

$$\begin{aligned} |A| &= a_{1\,3}A_{1\,3} + a_{2\,3}A_{2\,3} + a_{3\,3}A_{3\,3} + a_{4\,3}A_{4\,3} \\ &= (2)(6) + (0)A_{2\,3} + (0)A_{3\,3} + (1)(-10) \\ &= 12 - 10 = 2. \end{aligned}$$

(c) What is the $(3,1)$-entry of adj(A)?

Solution:

If $[A_{i\,j}]$ is the matrix of the cofactors of A, then adj$(A) = [A_{i\,j}]^t$. Consequently, the $(3,1)$-entry of adj(A) is the $(1,3)$-entry of $[A_{i\,j}]$.

In other words, the $(3,1)$-entry of adj(A) is actually the cofactor $A_{1\,3}$, which is 6.

(d) What is the $(3,4)$-entry of A^{-1}?

Solution:

Since $A^{-1} = \frac{1}{|A|}\text{adj}(A)$, we can find the $(3,4)$-entry of A^{-1} by finding the $(3,4)$-entry of $\text{adj}(A)$ and dividing it by $|A|$. Since the $(3,4)$-entry of $\text{adj}(A)$ is the cofactor $A_{4\,3}$, the $(3,4)$-entry of A^{-1} is $\frac{A_{4\,3}}{|A|} = \frac{-10}{2} = -5$.

6. Use each technique below to solve the system $\begin{cases} x + y = 1 \\ 3x + 2y = -1 \end{cases}$

(a) Form the augmented matrix and find its reduced echelon form.

Solution:

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 3 & 2 & -1 \end{array}\right] \underset{-3\text{R1}+\text{R2}}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & -1 & -4 \end{array}\right] \underset{-\text{R2}}{\sim} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 4 \end{array}\right] \underset{-\text{R2}+\text{R1}}{\sim} \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 4 \end{array}\right].$$

Ergo, the solution is $x = -3$ and $y = 4$.

(b) Form the matrix equation $AX = B$ and use A^{-1} to find X.

Solution:

Let $A = \begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, and $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Then, $$AX = B$$

$$\implies \quad A^{-1}(AX) = A^{-1}B$$

$$\implies \quad \left(A^{-1}A\right)X = \tfrac{1}{(1)(2)-(1)(3)}\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\implies \quad (I)X = \tfrac{1}{-1}\begin{bmatrix} 2 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\implies \quad X = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\implies \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}.$$

As expected, the solution is $x = -3$ and $y = 4$.

(c) Use Cramer's rule.

Solution:

$$x = \frac{\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}} = \frac{(1)(2)-(1)(-1)}{(1)(2)-(1)(3)} = \frac{3}{-1} = -3, \quad \text{and}$$

$$y = \frac{\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix}} = \frac{(1)(-1)-(1)(3)}{(1)(2)-(1)(3)} = \frac{-4}{-1} = 4, \quad \text{as expected.}$$