Math 1410–Solutions for the Final Exam Practice Sheet

The Final Exam will be 9:00am–11:00am, Friday, December 16, in PE250

1. Let $A = \begin{bmatrix} 0 & 1 & -5 & 4 \\ 2 & 1 & -9 & -10 \\ -2 & 0 & 4 & 6 \end{bmatrix}$. Find a basis and the dimension of

   (a) the row space of A,

Solution:

We need to find the reduced echelon form of A.

$$\begin{bmatrix} 0 & 1 & -5 & 4 \\ 2 & 1 & -9 & -10 \\ -2 & 0 & 4 & 6 \end{bmatrix}$$

$$\underset{\mathsf{R1} \longleftrightarrow \mathsf{R3}}{\sim} \begin{bmatrix} -2 & 0 & 4 & 6 \\ 2 & 1 & -9 & -10 \\ 0 & 1 & -5 & 4 \end{bmatrix}$$

$$\underset{-\frac{1}{2}\mathsf{R1}}{\sim} \begin{bmatrix} 1 & 0 & -2 & -3 \\ 2 & 1 & -9 & -10 \\ 0 & 1 & -5 & 4 \end{bmatrix}$$

$$\underset{-2\mathsf{R1}+\mathsf{R2}}{\sim} \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -5 & -4 \\ 0 & 1 & -5 & 4 \end{bmatrix}$$

$$\underset{-\mathsf{R2}+\mathsf{R3}}{\sim} \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 8 \end{bmatrix}$$

$$\underset{\frac{1}{8}\mathsf{R3}}{\sim} \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -5 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\substack{3\mathsf{R3}+\mathsf{R1} \\ 4\mathsf{R3}+\mathsf{R1}}}{\sim} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

To form a basis of the row space of A, we can use the non-zero rows of the reduced echelon form of A i.e.

$$\{(1,\ 0, -2,\ 0),\ (0,\ 1, -5,\ 0),\ (0,\ 0,\ 0,\ 1)\}$$

is a basis of row(A). Alternately, if we do not switch any rows, we can use the rows in A corresponding to the rows in which the leading ones appear in the reduced form of A. i.e.

$$\{(0,\ 1, -5,\ 4),\ (2,\ 1, -9, -10),\ (-2,\ 0,\ 4,\ 6)\}$$

is also a basis of row(A).

Lastly, the dimension of a space is the number of vectors in its basis, so the dimension of row(A) is 3.

(b) the solution set of $A\underline{x} = 0$.

<u>Solution</u>:

We use the work done in A. First, let $\underline{x} = \begin{bmatrix} w & x & y & z \end{bmatrix}^t$. Then, the general solution of $A\underline{x} = 0$ is

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2y \\ 5y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} 2 \\ 5 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore, $\{(2,\ 5,\ 1,\ 0)\}$ is a basis of the solution set of $A\underline{x} = 0$, and the dimension of this solution set is 1.

2. Show that $\underline{u}_1 = \left(\frac{3}{13}, \frac{4}{13}, \frac{12}{13}\right)$, $\underline{u}_2 = \left(\frac{4}{5}, -\frac{3}{5}, 0\right)$, and $\underline{u}_3 = \left(\frac{36}{65}, \frac{48}{65}, -\frac{25}{65}\right)$ form an orthonormal basis for $\mathbb{R}^3$.

<u>Solution</u>:

$$\underline{u}_1 \circ \underline{u}_2 = \left(\frac{3}{13}\right)\left(\frac{4}{5}\right) + \left(\frac{4}{13}\right)\left(-\frac{3}{5}\right) + 0 = \frac{12}{65} - \frac{12}{65} = 0,$$

$$\underline{u}_1 \circ \underline{u}_3 = \frac{3}{13} \cdot \frac{36}{65} + \frac{4}{13} \cdot \frac{48}{65} + \left(\frac{12}{13}\right)\left(-\frac{25}{65}\right) = \frac{108}{845} + \frac{192}{845} - \frac{300}{845} = 0,$$

$$\underline{u}_2 \circ \underline{u}_3 = \left(\frac{4}{5}\right)\left(\frac{36}{65}\right) + \left(-\frac{3}{5}\right)\left(\frac{48}{65}\right) + 0 = \frac{144}{325} - \frac{144}{325} = 0,$$

$$||\underline{u}_1|| = \sqrt{\left(\frac{3}{13}\right)^2 + \left(\frac{4}{13}\right)^2 + \left(\frac{12}{13}\right)^2} = \sqrt{\frac{9}{169} + \frac{16}{169} + \frac{144}{169}} = \sqrt{\frac{169}{169}} = 1,$$

$$||\underline{u}_2|| = \sqrt{\left(\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 + (0)^2} = \sqrt{\frac{16}{25} + \frac{9}{25}} = \sqrt{\frac{25}{25}} = 1, \text{ and}$$

$$||\underline{u}_3|| = \sqrt{\frac{36^2}{65^2} + \frac{48^2}{65^2} + \frac{25^2}{65^2}} = \sqrt{\frac{1296}{4225} + \frac{2304}{4225} + \frac{625}{4225}} = \sqrt{\frac{4225}{4225}} = 1,$$

so $\underline{u}_1$, $\underline{u}_2$, and $\underline{u}_3$ form an orthonormal basis for $\mathbb{R}^3$.

3. Let $\underline{v}_1 = (1, 1, 1, 1)$, $\underline{v}_2 = (3, 1, 9, -5)$, and $\underline{v}_3 = (13, -3, 11, -1)$.

   (a) Use Gram-Schmidt to orthonormalize $\underline{v}_1$, $\underline{v}_2$, and $\underline{v}_3$.

   Solution:

$$\underline{u}_1 = \underline{v}_1 = (1, 1, 1, 1),$$

$$\begin{aligned}
\underline{u}_2 &= \underline{v}_2 - \text{proj}_{\underline{u}_1}\underline{v}_2 = \underline{v}_2 - \tfrac{\underline{v}_2 \circ \underline{u}_1}{\underline{u}_1 \circ \underline{u}_1}\,\underline{u}_1 \\
&= (3, 1, 9, -5) - \tfrac{3+1+9-5}{1+1+1+1}(1, 1, 1, 1) \\
&= (3, 1, 9, -5) - \tfrac{8}{4}(1, 1, 1, 1) \\
&= (3, 1, 9, -5) - (2, 2, 2, 2) \\
&= (1, -1, 7, -7), \text{ and}
\end{aligned}$$

$$\begin{aligned}
\underline{u}_3 &= \underline{v}_3 - \text{proj}_{\underline{u}_1}\underline{v}_3 - \text{proj}_{\underline{u}_2}\underline{v}_3 \\
&= \underline{v}_3 - \tfrac{\underline{v}_3 \circ \underline{u}_1}{\underline{u}_1 \circ \underline{u}_1}\,\underline{u}_1 - \tfrac{\underline{v}_3 \circ \underline{u}_2}{\underline{u}_2 \circ \underline{u}_2}\,\underline{u}_2 \\
&= (13, -3, 11, -1) - \tfrac{13-3+11-1}{1+1+1+1}(1, 1, 1, 1) - \tfrac{13+3+77+7}{1+1+49+49}(1, -1, 7, -7) \\
&= (13, -3, 11, -1) - \tfrac{20}{4}(1, 1, 1, 1) - \tfrac{100}{100}(1, -1, 7, -7) \\
&= (1, 2, 2, 0) - (5, 5, 5, 5) - (1, -1, 7, -7) \\
&= (7, -7, -1, 1).
\end{aligned}$$

We now normalize these vectors to create the orthonormal basis $\{\underline{w}_1,\ \underline{w}_2,\ \underline{w}_3\}$:

$$\underline{w}_1 \ = \ \frac{1}{||\underline{u}_1||}\,\underline{u}_1 \ = \ \frac{1}{\sqrt{4}}\,(1,\ 1,\ 1,\ 1) \ = \ \left(\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}\right),$$

$$\underline{w}_2 \ = \ \frac{1}{||\underline{u}_2||}\,\underline{u}_2 \ = \ \frac{1}{\sqrt{100}}\,(1,-1,\ 7,-7) \ = \ \left(\frac{1}{10},-\frac{1}{10},\ \frac{7}{10},-\frac{7}{10}\right),$$

and $\underline{w}_3 \ = \ \frac{1}{||\underline{u}_3||}\,\underline{u}_3 \ = \ \frac{1}{\sqrt{100}}\,(7,-7,-1,\ 1) \ = \ \left(\frac{7}{10},-\frac{7}{10},-\frac{1}{10},\ \frac{1}{10}\right).$

(b) Determine whether $\underline{a} = (8,\ 8,\ 0,\ 0)$ is in the span of $\{\underline{v}_1,\ \underline{v}_2,\ \underline{v}_3\}$.

Solution:

$$\begin{aligned}
&\text{proj}_{\underline{w}_1}\underline{a} \ + \ \text{proj}_{\underline{w}_2}\underline{a} \ + \ \text{proj}_{\underline{w}_3}\underline{a} \\
= \ & (\underline{a}\circ\underline{w}_1)\ \underline{w}_1 \ + \ (\underline{a}\circ\underline{w}_2)\ \underline{w}_2 \ + \ (\underline{a}\circ\underline{w}_3)\ \underline{w}_3 \\
= \ & 8\left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right) \ + \ 0\left(\tfrac{1}{10},-\tfrac{1}{10},\ \tfrac{7}{10},-\tfrac{7}{10}\right) \ + \ 0\left(\tfrac{7}{10},-\tfrac{7}{10},-\tfrac{1}{10},\ \tfrac{1}{10}\right) \\
= \ & (4,\ 4,\ 4,\ 4) \ + \ 0 \ + \ 0 \ \neq \ \underline{a},
\end{aligned}$$

so $\underline{a}$ is not in the span of $\{\underline{v}_1,\ \underline{v}_2,\ \underline{v}_3\}$.

(c) Determine whether $\underline{b} = (30,-30,-40,\ 40)$ is in span$\{\underline{v}_1,\ \underline{v}_2,\ \underline{v}_3\}$.

Solution:

$$\begin{aligned}
&\text{proj}_{\underline{w}_1}\underline{b} \ + \ \text{proj}_{\underline{w}_2}\underline{b} \ + \ \text{proj}_{\underline{w}_3}\underline{b} \\
= \ & (\underline{b}\circ\underline{w}_1)\ \underline{w}_1 \ + \ (\underline{b}\circ\underline{w}_2)\ \underline{w}_2 \ + \ (\underline{b}\circ\underline{w}_3)\ \underline{w}_3 \\
= \ & 0\left(\tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2},\ \tfrac{1}{2}\right) \ + \ (-50)\left(\tfrac{1}{10},-\tfrac{1}{10},\ \tfrac{7}{10},-\tfrac{7}{10}\right) \ + \ (50)\left(\tfrac{7}{10},-\tfrac{7}{10},-\tfrac{1}{10},\ \tfrac{1}{10}\right) \\
= \ & 0 \ + \ (-5,\ 5,-35,\ 35) \ + \ (35,-35,-5,\ 5) \\
= \ & (30,-30,-40,\ 40) \ = \ \underline{b},
\end{aligned}$$

so $\underline{b}$ is in span$\{\underline{v}_1,\ \underline{v}_2,\ \underline{v}_3\}$.

4. Let $A = \begin{bmatrix} -1 & -3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 12 & 0 \\ 12 & -1 & 0 \\ 0 & 0 & 15 \end{bmatrix}$.

(a) Show that $(1, -1, 0)$ is an eigenvector of A and find the corresponding eigenvalue.

<u>Solution</u>:

We need to show that $\underline{x} = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^t$ is a non-zero solution of $A\underline{x} = \lambda\underline{x}$ for some value of the scalar λ.

$$A\underline{x} = \begin{bmatrix} -1 & -3 & 0 \\ 3 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = 2\underline{x}.$$

Therefore, $(1, -1, 0)$ is an eigenvector of A, and its corresponding eigenvalue is 2.

(b) Find all the eigenvalues of A.

<u>Solution</u>:

$$\begin{aligned} & |A - \lambda I| \\ = & \begin{vmatrix} -1-\lambda & -3 & 0 \\ 3 & 5-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} \\ = & (2-\lambda) \begin{vmatrix} -1-\lambda & -3 \\ 3 & 5-\lambda \end{vmatrix} \\ = & (2-\lambda)\left[(-1-\lambda)(5-\lambda) - (3)(-3)\right] \\ = & (2-\lambda)\left[-5+\lambda-5\lambda+\lambda^2+9\right] \\ = & (2-\lambda)\left[\lambda^2-4\lambda+4\right] \\ = & (2-\lambda)\left[(\lambda-2)(\lambda-2)\right] \\ = & -(\lambda-2)^3. \end{aligned}$$

So, the only eigenvalue of A is 2.

(c) Find a basis for each eigenspace of A.

Solution:

The augmented matrix for the system $(A-2I)\underline{x} = 0$ is

$$\left[\begin{array}{ccc|c} -1-(2) & -3 & 0 & 0 \\ 3 & 5-(2) & 0 & 0 \\ 0 & 0 & 2-(2) & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} -3 & -3 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{\text{R1}+\text{R2}}{\sim} \left[\begin{array}{ccc|c} -3 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{-\frac{1}{3}\text{R1}}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution to this system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -y \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

so $\{(-1,\ 1,\ 0),\ (0,\ 0,\ 1)\}$ is a basis of the eigenspace.

(d) Is A diagonalizable?

Solution:

We do not have enough linearly independent eigenvectors of A to create an invertible 3×3 matrix P, so A is not diagonalizable.

(e) Find all the eigenvalues of B.

Solution:

$$
\begin{aligned}
& |B - \lambda I| \\
= & \begin{vmatrix} 6-\lambda & 12 & 0 \\ 12 & -1-\lambda & 0 \\ 0 & 0 & 15-\lambda \end{vmatrix} \\
= & (15-\lambda)\begin{vmatrix} 6-\lambda & 12 \\ 12 & -1-\lambda \end{vmatrix} \\
= & (15-\lambda)\left[(6-\lambda)(-1-\lambda)-(12)^2\right] \\
= & (15-\lambda)\left[-6-6\lambda+\lambda+\lambda^2-144\right] \\
= & (15-\lambda)\left[\lambda^2-5\lambda-150\right] \\
= & (15-\lambda)\left[(\lambda-15)(\lambda+10)\right] \\
= & -(\lambda-15)^2(\lambda+10).
\end{aligned}
$$

So, the eigenvalues of B are 15 and -10.

(f) Find a basis for each eigenspace of B.

Solution:

<u>$\lambda = 15$</u>: The augmented matrix for the system $(B-15I)\underline{x} = 0$ is

$$
\left[\begin{array}{ccc|c} 6-(15) & 12 & 0 & 0 \\ 12 & -1-(15) & 0 & 0 \\ 0 & 0 & 15-(15) & 0 \end{array}\right]
$$

$$
= \left[\begin{array}{ccc|c} -9 & 12 & 0 & 0 \\ 12 & -16 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]
$$

$$\underset{\substack{-\frac{1}{9}\text{R1} \\ \frac{1}{12}\text{R2}}}{\sim} \left[\begin{array}{ccc|c} 1 & -\frac{4}{3} & 0 & 0 \\ 1 & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\underset{-\text{R1}+\text{R2}}{\sim} \left[\begin{array}{ccc|c} 1 & -\frac{4}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution to this system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{4}{3}y \\ y \\ z \end{bmatrix} = y \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

so $\left\{ \left(\frac{4}{3},\ 1,\ 0 \right),\ (0,\ 0,\ 1) \right\}$ is a basis of this eigenspace.

$\underline{\lambda = -10}$: The augmented matrix for the system $(B - (-10)I)\underline{x} = 0$ is

$$\left[\begin{array}{ccc|c} 6-(-10) & 12 & 0 & 0 \\ 12 & -1-(-10) & 0 & 0 \\ 0 & 0 & 15-(-10) & 0 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 16 & 12 & 0 & 0 \\ 12 & 9 & 0 & 0 \\ 0 & 0 & 25 & 0 \end{array}\right]$$

$$\underset{\substack{\frac{1}{16}\text{R1} \\ \frac{1}{12}\text{R2} \\ \frac{1}{25}\text{R3}}}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{3}{4} & 0 & 0 \\ 1 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{-\text{R1}+\text{R2}}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\underset{\text{R2} \longleftrightarrow \text{R3}}{\sim} \left[\begin{array}{ccc|c} 1 & \frac{3}{4} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The general solution to this system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\frac{3}{4}y \\ y \\ 0 \end{bmatrix} = y \begin{bmatrix} -\frac{3}{4} \\ 1 \\ 0 \end{bmatrix},$$

so $\left\{\left(-\frac{3}{4},\ 1,\ 0\right)\right\}$ is a basis of this eigenspace.

(g) Orthonormalize the vectors found in (f), using Gram-Schmidt if necessary.

Solution:

The vectors $\left(\frac{4}{3},\ 1,\ 0\right)$, $(0,\ 0,\ 1)$, and $\left(-\frac{3}{4},\ 1,\ 0\right)$ are already orthogonal, so we just need to normalize them:

$$\frac{1}{\left|\left|\left(\frac{4}{3},\ 1,\ 0\right)\right|\right|} \left(\frac{4}{3},\ 1,\ 0\right) = \frac{1}{5/3} \left(\frac{4}{3},\ 1,\ 0\right) = \left(\frac{4}{5},\ \frac{3}{5},\ 0\right),$$

$$\frac{1}{||(0,\ 0,\ 1)||} (0,\ 0,\ 1) = \frac{1}{1} (0,\ 0,\ 1) = (0,\ 0,\ 1),$$

and $\dfrac{1}{\left|\left|\left(-\frac{3}{4},\,1,\,0\right)\right|\right|}\left(-\dfrac{3}{4},\,1,\,0\right) = \dfrac{1}{5/4}\left(-\dfrac{3}{4},\,1,\,0\right) = \left(-\dfrac{3}{5},\,\dfrac{4}{5},\,0\right).$

(h) Use the vectors found in (g) to create a matrix P that diagonalizes B.

Solution:

The columns of P are the vectors we found in (g):

$$P = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We now verify that P diagonalizes B:

$$\begin{aligned} P^tBP &= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} 6 & 12 & 0 \\ 12 & -1 & 0 \\ 0 & 0 & 15 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & 12 & 0 \\ -8 & 9 & 0 \\ 0 & 0 & 15 \end{bmatrix} \\ &= \begin{bmatrix} -10 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix} = D. \end{aligned}$$

D is a diagonal matrix, so P diagonalizes A.

(i) Find the (1,1)-entry in B^5.

Solution:

$$B^5 = PD^5P^t$$

$$= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -10 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \end{bmatrix}^5 \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}^t$$

$$= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -10^5 & 0 & 0 \\ 0 & 15^5 & 0 \\ 0 & 0 & 15^5 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{5}\left(10^5\right) & -\frac{4}{5}\left(10^5\right) & 0 \\ \frac{4}{5}\left(15^5\right) & \frac{3}{5}\left(15^5\right) & 0 \\ 0 & 0 & 15^5 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{9}{25}\left(10^5\right)+\frac{16}{25}\left(15^5\right) & \frac{12}{25}\left(10^5\right)+\frac{12}{25}\left(15^5\right) & 0 \\ \frac{12}{25}\left(10^5\right)+\frac{12}{25}\left(10^5\right) & -\frac{16}{25}\left(10^5\right)+\frac{9}{25}\left(10^5\right) & 0 \\ 0 & 0 & 15^5 \end{bmatrix}.$$

So, the (1,1)-entry of B^7 is $-\dfrac{9}{25}\left(10^5\right)+\dfrac{16}{25}\left(15^5\right)$, or 450000.

5. Determine whether each statement is true or false. Justify your answer.

   (a) $(br-cq,\ cp-ar,\ aq-bp)$ is orthogonal to both $(a,\ b,\ c)$ and $(p,\ q,\ r)$.

   Solution: This statement is TRUE.

$$\begin{aligned} & (br-cq,\ cp-ar,\ aq-bp) \circ (a,\ b,\ c) \\ = \ & (br-cq)a + (cp-ar)b + (aq-bp)c \\ = \ & bra - cqa + cpb - arb + aqc - bpc = 0, \text{ and} \end{aligned}$$

$$\begin{aligned} & (br-cq,\ cp-ar,\ aq-bp) \circ (p,\ q,\ r) \\ = \ & (br-cq)p + (cp-ar)q + (aq-bp)r \\ = \ & brp - cqp + cpq - arq + aqr - bpr = 0. \end{aligned}$$

(b) $\{(x,\ y) :\ x^2 = xy\}$ is a subspace of $\mathbb{R}^2$.

<u>Solution</u>: This statement is FALSE.

This subset of $\mathbb{R}^2$ contains $(0,\ 1)$ and $(1,\ 1)$, but it does not contain

$$(0,\ 1)\ +\ (1,\ 1)\ =\ (1,\ 2).$$

(c) If 0 is an eigenvalue of an $n \times n$ matrix A, then A is invertible.

<u>Solution</u>: This statement is FALSE.

Let 0 be an eigenvalue of A. Then, $\lambda = 0$ satisfies $|A - \lambda I| = 0$

$$\implies |A - 0I| = 0 \implies |A| = 0,$$

which means that A is not invertible.

(c) (Bonus) If 0 is an eigenvalue of an $n \times n$ matrix A, then A is not invertible.

<u>Solution</u>: This statement is TRUE.

Let 0 be an eigenvalue of A. Then, $\lambda = 0$ satisfies $|A - \lambda I| = 0$

$$\implies |A - 0I| = 0 \implies |A| = 0,$$

which means that A is not invertible.

(d) (Bonus) If $A^3 + 8I = 0$ and $A^t = A$, then A is both invertible *and* diagonalizable.

<u>Solution</u>: This statement is TRUE.

Let A be a matrix that satisfy $A^3 + 8I = 0$ and $A^t = A$. To show that A is invertible, we manipulate $A^3 + 8I = 0$

$$\implies A^3 = -8I \implies -\frac{1}{8}A^3 = I \implies A\left(-\frac{1}{8}A^2\right) = I.$$

Since A is square (necessarily), A is invertible, and $A^{-1} = -\dfrac{1}{8}A^2$. Next, if $A^t = A$, then A is symmetric, so by Theorem 5.4 on page 160 of the textbook, A must be diagonalizable.