# Math 1410–Solutions for Assignment 3

Submitted Friday, October 7

1. Find $2A - A^2$, where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Solution:**

$$\begin{aligned}
& 2A - A^2 \\
= \; & 2\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} \\
= \; & \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
= \; & \begin{pmatrix} 0 & 2 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}.
\end{aligned}$$

2. Find the values of $x$ and $y$ that make $AB + A = 0$, where

$$A = \begin{pmatrix} x & 1 \\ 1 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ y & 1 \end{pmatrix}.$$

**Solution:**

$$\begin{aligned}
& AB + A = 0 \\
\Longrightarrow \; & \begin{pmatrix} x & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ y & 1 \end{pmatrix} + \begin{pmatrix} x & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\Longrightarrow \; & \begin{pmatrix} x+y & 2x+1 \\ 1-y & 0 \end{pmatrix} + \begin{pmatrix} x & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\Longrightarrow \; & \begin{pmatrix} 2x+y & 2x+2 \\ 2-y & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

$$\Longrightarrow \quad \left\{ \begin{array}{rcl} 2x+y & = & 0 \\ 2x+2 & = & 0 \\ 2-y & = & 0 \\ 0 & = & 0 \end{array} \right.$$

$$\Longrightarrow \quad \left\{ \begin{array}{rcr} x & = & -1 \\ y & = & 2 \end{array} \right. .$$

3. Find $A^3 - 4A^2 + 4A$, where $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

**<u>Solution</u>:**

To begin,

$$A^2 = AA = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix},$$

so

$$A^3 = A^2A = \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Therefore,

$$\begin{aligned} & A^3 - 4A^2 + 4A \\ = \; & \begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} - 4 \begin{pmatrix} 2 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} + 4 \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \\ = \; & \begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} - \begin{pmatrix} 8 & 8 & 0 \\ 8 & 8 & 0 \\ 0 & 0 & 16 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} \\ = \; & \begin{pmatrix} -4 & -4 & 0 \\ -4 & -4 & 0 \\ 0 & 0 & -8 \end{pmatrix} + \begin{pmatrix} 4 & 4 & 0 \\ 4 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix} \\ = \; & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

4. Write the following system of equations in matrix form $AX = B$ and identify the matrices $A, X$ and $B$:

$$\begin{array}{rcrcrcrcr} 3x^2 & - & 2y & + & 3z^3 & + & w & = & 7 \\ 2x^2 & - & 3y & - & 4z^3 & - & w & = & 3 \\ 11x^2 & + & y & + & 4z^3 & - & w & = & 31 \\ x^2 & + & y & - & 2z^3 & + & w & = & 11 \end{array}$$

**<u>Solution</u>:**

Let

$$A = \begin{pmatrix} 3 & -2 & 3 & 1 \\ 2 & -3 & -4 & -1 \\ 11 & 1 & 4 & -1 \\ 1 & 1 & -2 & 1 \end{pmatrix},$$

$$X = \begin{pmatrix} x^2 \\ y \\ z^3 \\ w \end{pmatrix}, \text{ and } B = \begin{pmatrix} 7 \\ 3 \\ 31 \\ 11 \end{pmatrix}.$$

Then, the matrix form $AX = B$ of the system of equations above is

$$\begin{pmatrix} 3 & -2 & 3 & 1 \\ 2 & -3 & -4 & -1 \\ 11 & 1 & 4 & -1 \\ 1 & 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} x^2 \\ y \\ z^3 \\ w \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 31 \\ 11 \end{pmatrix}.$$

5. If $A$, $B$, and $C$ are any $2 \times 2$ matrices, prove or disprove each of the following statements:
(a) $AB - BA = 0$ for all $A$ and $B$.

**<u>Solution</u>:** This statement is FALSE.

Let $A = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}$. Then,

$$\begin{aligned} AB - BA &= \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 4 & 5 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 8 \\ 4 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 8 \\ 4 & 6 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \neq 0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

(b) $A^3 = 0$ implies $A = 0$.

**Solution:** This statement is FALSE.

Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then,

$$\begin{aligned} A^3 &= A^2 A \\ &= \left[\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \left[\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\right]\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

However, $A \neq 0$.

(c) $BC = 2AC$ implies $B = 2A$ for all $A$, $B$ and $C$.

**Solution:** This statement is FALSE.

Let $A = B = I$ and $C = 0$. Then, $BC = 0$, and $2AC = 0$, so $BC = 2AC$. However, $I \neq 2I$, so $B \neq 2A$.

(d) If $AB = BA$, then $(A-B)^2 = A^2 - 2AB + B^2$.

**Solution:** This statement is TRUE.

Let $A$ and $B$ be any $2 \times 2$ matrices such that $AB = BA$. Then,

$$\begin{aligned} (A-B)^2 &= (A-B)(A-B) \\ &= A(A-B) - B(A-B) \\ &= A^2 - AB - BA + B^2 \\ &= A^2 - AB - AB + B^2 \quad (\text{since } AB = BA) \\ &= A^2 - 2AB + B^2, \quad \text{as required.} \end{aligned}$$

6. Let

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & -1 & 1 \end{pmatrix}.$$

Find elementary matrices $E_1, E_2, \cdots, E_n$, such that $E_n \cdots E_2 E_1 A$ is the reduced echelon form of $A$ and write down the matrix $E_n \cdots E_2 E_1 A$.

**Solution:**

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & -1 & 1 \end{pmatrix}$$

$$\underset{-\text{R1}+\text{R2}}{\sim} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & -1 & 1 \end{pmatrix} \qquad E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{-\text{R1}+\text{R4}}{\sim} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{\text{R2} \longleftrightarrow \text{R3}}{\sim} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \qquad E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{-\text{R2}+\text{R4}}{\sim} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad E_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$$

$$\underset{-\text{R2}+\text{R1}}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad E_5 = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\underset{-\text{R3}+\text{R4}}{\sim} \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad E_6 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

$$\underset{\text{R3}+\text{R2}}{\sim} \quad \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \Bigg| \quad E_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Hence,

$$E_7E_6E_5E_4E_3E_2E_1A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

7. Let $A = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$, where $a$ is any number. Find a $2 \times 2$ matrix $B$ such that $BA = I$ and find $AB$.

**Solution:**

Let $B = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Then, $BA = I$

$$\Longrightarrow \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{pmatrix} w+ax & x \\ y+az & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Longrightarrow \begin{cases} w+ax &= 1 \\ x &= 0 \\ y+az &= 0 \\ z &= 1 \end{cases}$$

$$\Longrightarrow \begin{cases} w &= 1 \\ x &= 0 \\ y &= -a \\ z &= 1 \end{cases}.$$

Thus, $B = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$, so

$$AB = \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$