Math 1410–Solutions for Assignment 6

Submitted Friday, November 4

1. Given the vectors $\underline{a} = (1,1,1)$, $\underline{b} = (-2,-1,2)$, and $\underline{v} = (3,4,7)$. Express the vector $\underline{v}$ as a linear combination of $\underline{a}$ and $\underline{b}$.

   Solution:

   We want to find scalars c_1 and c_2 such that $\underline{v} = c_1\underline{a} + c_2\underline{b}$, or

   $$c_1(1,1,1) + c_2(-2,-1,2) = (3,4,7).$$

   Focusing on the first component of each vector (i.e. the x coordinate), we have $c_1(1) + c_2(-2) = (3)$, or $c_1 - 2c_2 = 3$, which is a linear equation. Focusing on the second and third components of each vector gives us two more linear equations, which together form the following linear system:

   $$\begin{cases} c_1 - 2c_2 = 3 \\ c_1 - c_2 = 4 \\ c_1 + 2c_2 = 7 \end{cases}.$$

   To solve this system, we form its augmented matrix:

   $$\left[\begin{array}{rr|r} 1 & -2 & 3 \\ 1 & -1 & 4 \\ 1 & 2 & 7 \end{array}\right].$$

   Notice that the entries in the first three columns of this augmented matrix are the components of the vectors $\underline{a}$, $\underline{b}$, and $\underline{v}$ respectively. Now we need to find the reduced echelon form of this matrix:

   $$\begin{array}{c} \\ -\mathrm{R1}+\mathrm{R2} \\ -\mathrm{R1}+\mathrm{R3} \end{array} \left[\begin{array}{rr|r} 1 & -2 & 3 \\ 0 & 1 & 1 \\ 0 & 4 & 4 \end{array}\right]$$

$$\begin{array}{c} \sim \\ -2\mathrm{R}2+\mathrm{R}1 \\ -4\mathrm{R}2+\mathrm{R}3 \end{array} \quad \left[\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right].$$

The solution is $c_1 = 5$ and $c_2 = 1$. Therefore, $\underline{v} = 5\underline{a} + 2\underline{b}$.

2. Show that the *span*$\{(-1,0,1),(-2,0,-2)\}$ consists of all three dimensional vectors with the second component 0.

Solution:

We need to show that every vector of the form $(x,0,z)$, where x and z are real, is in *span*$\{(-1,0,1),(-2,0,-2)\}$. A vector is in the span of $(-1,0,1)$ and $(-2,0,-2)$ if it is a linear combination of $(-1,0,1)$ and $(-2,0,-2)$. So, we need to show that every vector of the form $(x,0,z)$ is a linear combination of $(-1,0,1)$ and $(-2,0,-2)$. To do this, we solve the following system (shown as an augmented matrix) for c_1 and c_2:

$$\left[\begin{array}{rr|c} -1 & -2 & x \\ 0 & 0 & 0 \\ 1 & -2 & z \end{array}\right]$$

$$\begin{array}{c} \sim \\ -\mathrm{R}1 \end{array} \quad \left[\begin{array}{rr|r} 1 & 2 & -x \\ 0 & 0 & 0 \\ 1 & -2 & z \end{array}\right]$$

$$\begin{array}{c} \sim \\ \mathrm{R}2 \longleftrightarrow \mathrm{R}3 \end{array} \quad \left[\begin{array}{rr|r} 1 & 2 & -x \\ 1 & -2 & z \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \sim \\ -\mathrm{R}1+\mathrm{R}2 \end{array} \quad \left[\begin{array}{rr|c} 1 & 2 & -x \\ 0 & -4 & x+z \\ 0 & 0 & 0 \end{array}\right]$$

$$
\underset{-\frac{1}{4}\mathsf{R2}}{\sim}\quad
\left[\begin{array}{cc|c}
1 & 2 & -x \\
0 & 1 & -\frac{x+z}{4} \\
0 & 0 & 0
\end{array}\right]
$$

$$
\underset{-2\mathsf{R2}+\mathsf{R1}}{\sim}\quad
\left[\begin{array}{cc|c}
1 & 0 & \frac{z-x}{2} \\
0 & 1 & -\frac{x+z}{4} \\
0 & 0 & 0
\end{array}\right].
$$

For any real x and z, there is a solution for c_1 and c_2. Thus, every vector of the form $(x,0,z)$ is in the span of $(-1,0,1)$ and $(-2,0,-2)$. Hence, *span*$\{(-1,0,1),(-2,0,-2)\}$ consists of all three dimensional vectors with the second component 0.

3. Given $S=\{(1,0,1,1),(0,1,1,1),(1,1,0,1)\}$,

   (a) determine whether the vector $\underline{v} = (-2,-3,3,-1)$ is in the span of S,

   Solution:

   Again, $\underline{v}$ is in the span of S if it is a linear combination of the vectors in S. To determine if it is, we solve the following system for c_1, c_2, and c_3:

$$
\left[\begin{array}{ccc|c}
1 & 0 & 1 & -2 \\
0 & 1 & 1 & -3 \\
1 & 1 & 0 & 3 \\
1 & 1 & 1 & -1
\end{array}\right]
$$

$$
\underset{\substack{-\mathsf{R1}+\mathsf{R3}\\ -\mathsf{R1}+\mathsf{R4}}}{\sim}\quad
\left[\begin{array}{ccc|c}
1 & 0 & 1 & -2 \\
0 & 1 & 1 & -3 \\
0 & 1 & -1 & 5 \\
0 & 1 & 0 & 1
\end{array}\right]
$$

$$\underset{\substack{-\mathsf{R2}+\mathsf{R3}\\-\mathsf{R2}+\mathsf{R4}}}{\sim}\left[\begin{array}{ccr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & -2 & 8 \\ 0 & 0 & -1 & 4 \end{array}\right]$$

$$\underset{-\frac{1}{2}\mathsf{R3}}{\sim}\left[\begin{array}{ccr|r} 1 & 0 & 1 & -2 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -1 & 4 \end{array}\right]$$

$$\underset{\substack{-\mathsf{R3}+\mathsf{R1}\\-\mathsf{R3}+\mathsf{R2}\\\mathsf{R3}+\mathsf{R4}}}{\sim}\left[\begin{array}{ccc|r} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The system is consistent, so $\underline{v}$ is a linear combination of the vectors in S. Ergo, $\underline{v}$ in the span of S.

(b) and if the vector is in the span, write it as a linear combination of elements of S.

Solution:

From the work above, we see that $c_1 = 2$, $c_2 = 1$, and $c_3 = -4$. Hence,

$$\underline{v} = 2\,(1,0,1,1) + 1\,(0,1,1,1) - 4\,(1,1,0,1)\,.$$

4. Which of the following subsets are subspaces of $\mathbb{R}^3$? You need to justify your claim.

(a) $\{(x,0,z): \ x = 3z\}$

Solution:

Every vector in this set has the form $(3z,\ 0,\ z)$. To determine if this set is a subspace of $\mathbb{R}^3$, we need to determine if it is closed under vector addition and scalar multiplication.

To check closure under vector addition, let $\underline{v}_1 = (3z_1,\ 0,\ z_1)$ and $\underline{v}_2 = (3z_2,\ 0,\ z_2)$ be any vectors in the set. Then,

$$\begin{aligned}\underline{v}_1 + \underline{v}_2 &= (3z_1,\ 0,\ z_1) + (3z_2,\ 0,\ z_2) \\ &= (3z_1 + 3z_2,\ 0,\ z_1 + z_2) \\ &= (3(z_1 + z_2),\ 0,\ (z_1 + z_2)),\end{aligned}$$

so $\underline{v}_1 + \underline{v}_2$ is also in the set. Consequently, the set is closed under vector addition.

To check closure under scalar multiplication, let $\underline{v}_0 = (3z_0,\ 0,\ z_0)$ be any vector in the set and let s be any scalar. Then,

$$\begin{aligned}s\underline{v}_0 &= s(3z_0,\ 0,\ z_0) = (s(3z_0),\ 0,\ s(z_0)) \\ &= (3sz_0,\ 0,\ sz_0) = (3(sz_0),\ 0,\ (sz_0)),\end{aligned}$$

so $s\underline{v}$ is also in the set. Thus, the set is closed under scalar multiplication. Since it is closed under both vector addition and scalar multiplication, this set is a subspace of $\mathbb{R}^3$.

(b) $\{(x,y,z) :\ \ x + 3y - 2z = 5\}$

Solution:

Let $\underline{v}$ be any vector in this set. In order for this set to be a subspace of $\mathbb{R}^3$, it must be closed under scalar multiplication. Thus, this set must contain every scalar multiple of $\underline{v}$, including $0\underline{v} = \underline{0} = (0,0,0)$. However, $(0,0,0)$ does not satisfy $x + 3y - 2z = 5$, so $(0,0,0)$ is not in the set.

Therefore, this set is NOT a subspace of $\mathbb{R}^3$.

(c) $\{(x,y,z): \;\; 3x+3y-z=0\}$

Solution:

Solving the equation $3x+3y-z=0$ for z, we find that every vector in this set has the form $(x,\ y,\ 3x+3y)$. As before, to determine if this set is a subspace of $\mathbb{R}^3$, we need to determine if it is closed under vector addition and scalar multiplication.

To check closure under vector addition, let $\underline{\text{v}}_1 = (x_1,\ y_1,\ 3x_1+3y_1)$ and $\underline{\text{v}}_2 = (x_2,\ y_2,\ 3x_2+3y_2)$ be any vectors in the set. Then,

$$\begin{aligned}
\underline{\text{v}}_1 + \underline{\text{v}}_2 &= (x_1,\ y_1,\ 3x_1+3y_1) + (x_2,\ y_2,\ 3x_2+3y_2) \\
&= (x_1+x_2,\ y_1+y_2,\ 3x_1+3y_1+3x_2+3y_2) \\
&= ((x_1+x_2),\ (y_1+y_2),\ 3(x_1+x_2)+3(y_1+y_2)).
\end{aligned}$$

Since $\underline{\text{v}}_1 + \underline{\text{v}}_2$ is also in the set, the set is closed under vector addition.

To check closure under scalar multiplication, let $\underline{\text{v}}_0 = (x_0,\ y_0,\ 3x_0+3y_0)$ be any vector in the set and let s be any scalar. Then,

$$\begin{aligned}
s\underline{\text{v}}_0 &= s(x_0,\ y_0,\ 3x_0+3y_0) = (sx_0,\ sy_0,\ s(3x_0+3y_0)) \\
&= (sx_0,\ sy_0,\ 3sx_0+3sy_0) \\
&= ((sx_0),\ (sy_0),\ 3(sx_0)+3(sy_0)).
\end{aligned}$$

Since $s\underline{\text{v}}$ is also in the set, the set is closed under scalar multiplication. Since the set is closed under both vector addition and scalar multiplication, it is a subspace of $\mathbb{R}^3$.

(d) $\{(x,y,z): \ \ xy+z=0\}$

Solution:

If this set is a subspace of $\mathbb{R}^3$, then it must be closed under vector addition. However, the vectors $(1,0,0)$ and $(0,1,0)$ are in the set, but their sum, $(1,0,0)+(0,1,0)=(1,1,0)$, is not.

Since this set is not closed under vector addition, it is NOT a subspace of $\mathbb{R}^3$.

(e) $\{(x,y,z): \ \ x^2=0\}$

Solution:

If $x^2=0$, then $x=0$, so every vector in this set has the form $(0,\ y,\ z)$.

To check if this set is closed under vector addition, let $\underline{v}_1=(0,\ y_1,\ z_1)$ and $\underline{v}_2=(0,\ y_2,\ z_2)$ be any vectors in the set. Then,

$$\begin{aligned}\underline{v}_1+\underline{v}_2 &= (0,\ y_1,\ z_1)+(0,\ y_2,\ z_2)\\ &= (0,\ y_1+y_2,\ z_1+z_2)\,,\end{aligned}$$

which is also in the set, so the set is closed under vector addition.

To check closure under scalar multiplication, let $\underline{v}_0=(0,\ y_0,\ z_0)$ be any vector in the set and let s be any scalar. Then,

$$s\underline{v}_0=s\,(0,\ y_0,\ z_0)=(0,\ sy_0,\ sz_0)\,,$$

which is also in the set, so the set is closed under scalar multiplication. Since the set is closed under both vector addition and scalar multiplication, it is a subspace of $\mathbb{R}^3$.

(f) $\{(x,y,z): \ x+3y-z=0 \text{ and } 2x-3y+z=0\}$

Solution:

Solving the linear system formed by the equations $x+3y-z=0$ and $2x-3y+z=0$, we find that $x=0$ and $z=3y$, so every vector in the set has the form $(0,\ y,\ 3y)$.

To check if this set is closed under vector addition, let $\underline{v}_1=(0,\ y_1,\ 3y_1)$ and $\underline{v}_2=(0,\ y_2,\ 3y_2)$ be any vectors in the set. Then,

$$\begin{aligned}\underline{v}_1+\underline{v}_2 &= (0,\ y_1,\ 3y_1)+(0,\ y_2,\ 3y_2)\\ &= (0,\ y_1+y_2,\ 3y_1+3y_2)\\ &= (0,\ (y_1+y_2),\ 3(y_1+y_2)),\end{aligned}$$

which is also in the set, so the set is closed under vector addition.

To check closure under scalar multiplication, let $\underline{v}_0=(0,\ y_0,\ 3y_0)$ be any vector in the set and let s be any scalar. Then,

$$\begin{aligned}s\underline{v}_0 &= s(0,\ y_0,\ 3y_0)=(0,\ sy_0,\ s(3y_0))\\ &= (0,\ (sy_0),\ 3(sy_0)),\end{aligned}$$

which is also in the set, so the set is closed under scalar multiplication. Since the set is closed under both vector addition and scalar multiplication, it is a subspace of $\mathbb{R}^3$.

5. Consider the system of the linear equations $AX=0$, where A is any $n\times m$ matrix and X a column vector of dimension m. Show that

$$S=\{X:\ AX=0\},$$

the solution set of the system, is a subspace of $\mathbb{R}^m$.

Solution:

To show that S is a subspace of $\mathbb{R}^m$, we need to show that it is closed under vector addition and scalar multiplication. To show closure under vector addition, let X_1 and X_2 be any column vectors in S. Then, $AX_1 = 0$ and $AX_2 = 0$, so

$$A(X_1 + X_2) = AX_1 + AX_2 = 0 + 0 = 0,$$

which shows that $X_1 + X_2$ is also a solution of $AX = 0$. In other words, $X_1 + X_2$ is also in S, so S is closed under vector addition.

To show closure under scalar multiplication, let X_0 be any column vector in S and let c be any scalar. Then, $AX_0 = 0$, so

$$A(cX_0) = cAX_0 = c(AX_0) = c(0) = 0,$$

which shows that cX_0 is also a solution of $AX = 0$. In other words, cX_0 is also in S, so S is closed under scalar multiplication.

Since S is closed under vector addition and scalar multiplication, it is a subspace of $\mathbb{R}^m$.

6. (Bonus) A plane in $\mathbb{R}^3$ is the set of all vectors (x, y, z) satisfying an equation like $ax + by + cz = d$. Show that a plane is a subspace if and only if it contains the origin.

Solution:

Since this is an "if and only if" statement, we need to prove it in both directions.

$(\Longrightarrow)$ Assume that the plane is a subspace of $\mathbb{R}^3$, and pick any vector $\underline{v}$ in the plane. Then, the plane must be closed under scalar multiplication, so it must contain every scalar multiple of $\underline{v}$, including $0\underline{v} = \underline{0} = (0, 0, 0)$. In other words, the plane must contain the origin.

$(\Longleftarrow)$ Assume that the plane contains the origin. Then, $(0,0,0)$ must satisfy the equation $ax+by+cz=d$, which means that $d=0$.

Now, let $\underline{v}_1=(x_1,\ y_1,\ z_1)$ and $\underline{v}_2=(x_2,\ y_2,\ z_2)$ be any vectors in the plane. Then, $ax_1+by_1+cz_1=0$ and $ax_2+by_2+cz_2=0$, so

$$
\begin{aligned}
& a\,(x_1+x_2)+b\,(y_1+y_2)+c\,(z_1+z_2) \\
= \;& ax_1+ax_2+by_1+by_2+cz_1+cz_2 \\
= \;& (ax_1+by_1+cz_1)+(ax_2+by_2+cz_2) \\
= \;& (0)+(0)=0,
\end{aligned}
$$

so the plane contains $\underline{v}_1+\underline{v}_2$. Thus, the plane is closed under vector addition.

Next, let $\underline{v}_0=(x_0,\ y_0,\ z_0)$ be any vector in the plane and let s be any scalar. Then, $ax_0+by_0+cz_0=0$, so

$$
\begin{aligned}
a\,(sx_0)+b\,(sy_0)+c\,(sz_0) &= asx_0+bsy_0+csz_0 \\
&= s\,(ax_0+by_0+cz_0)=s\,(0)=0,
\end{aligned}
$$

so the plane contains $s\underline{v}_0$. Hence, the plane is closed under scalar multiplication.

Ergo, the plane is a subspace of $\mathbb{R}^3$.