Math 1410–Solutions for Assignment 7

Submitted Friday, November 18

1. Determine if the following two sets of vectors span the same vector space:
   $S = \{(1,1,1,2),(0,1,1,1),(1,0,0,1)\}$, $T = \{(-1,1,1,0),(2,1,1,3),(1,2,2,3)\}$.

   Solution:

   Let A be a 3×4 matrix whose rows are the vectors from S

$$\text{i.e. } A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

   Then, the span of S is the same as the row space of A. Similarly, let B be a 3×4 matrix whose rows are the vectors from T

$$\text{i.e. } B = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 2 & 3 \end{bmatrix}.$$

   So, S and T will have the same span if and only if A and B have the same row space. Two matrices have same row space if and only if their reduced echelon forms have the same nonzero rows. Accordingly, we need to find the reduced echelon forms of A and B. Beginning with A,

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\underset{-\mathsf{R1}+\mathsf{R3}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & -1 & -1 \end{bmatrix}$$

$$\underset{\substack{-\text{R2}+\text{R1} \\ \text{R2}+\text{R3}}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Next, we reduce $B = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 2 & 3 \end{bmatrix}$

$$\underset{-\text{R1}}{\sim} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 2 & 1 & 1 & 3 \\ 1 & 2 & 2 & 3 \end{bmatrix}$$

$$\underset{\substack{-2\text{R1}+\text{R2} \\ -\text{R1}+\text{R3}}}{\sim} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 3 & 3 & 3 \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

$$\underset{\frac{1}{3}\text{R2}}{\sim} \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \end{bmatrix}$$

$$\underset{\substack{\text{R2}+\text{R1} \\ -3\text{R2}+\text{R3}}}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The reduced echelon forms of A and B have the same nonzero rows, so they have the same row space. Thus, S and T have the same span.

2. Let A be an invertible 3×3 matrix.

   (a) Show that the reduced echelon form of A is the identity matrix of order 3.

   <u>Solution</u>:

   If A is invertible, then the reduced echelon form of $\left[\begin{array}{c|c} A & I \end{array}\right]$ must be $\left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$ by the matrix inversion algorithm. Hence, the reduced echelon form of A must be I.

Alternately, consider the system $AX = 0$, where $X = \begin{bmatrix} x & y & z \end{bmatrix}^t$. Since A is invertible, A^{-1} exists, so we have $A^{-1}(AX) = A^{-1}(0)$, or $X = 0$. Thus, the system has a unique solution: $x = 0$, $y = 0$, and $z = 0$.

If we had solved this system by finding the reduced echelon form of $\left[\begin{array}{c|c} A & 0 \end{array}\right]$, we would have arrived at the same result. In other words, the reduced echelon form of $\left[\begin{array}{c|c} A & 0 \end{array}\right]$ must be $\left[\begin{array}{c|c} I & 0 \end{array}\right]$. Ergo, the reduced echelon form of A must be I.

(b) Explain why the row space of A is all of $\mathbb{R}^3$.

<u>Solution</u>:

Since A is equivalent to I, A and I have the same row space. The row space of I is the span of the set $\{(1,0,0),\ (0,1,0),\ (0,0,1)\}$. Since

$$(x,y,z) = x(1,0,0) + y(0,1,0) + z(0,0,1)$$

for all x, y, and z, the span of this set is all of $\mathbb{R}^3$. Therefore, the row space of A is all of $\mathbb{R}^3$.

3. Given that $S = \{(-1,1,1,1),(-1,-1,1,1),(-1,-1,-1,1)\}$. Determine which of the vectors $\underline{v} = (2,3,4,-2)$, $\underline{u} = (1,2,3,4)$, are in the span of S?

<u>Solution</u>:

We need to determine if there are scalars s_1, s_2, and s_3 such that

$$s_1(-1,1,1,1) + s_2((-1,-1,1,1)) + s_3(-1,-1,-1,1)$$

is equal to $\underline{v}$ or $\underline{u}$. Rather than solving the two systems separately, we can save some time by augmenting the coefficient matrix with *two* solution columns, as follows:

$$\left[\begin{array}{rrr|rr} -1 & -1 & -1 & 2 & 1 \\ 1 & -1 & -1 & 3 & 2 \\ 1 & 1 & -1 & 4 & 3 \\ 1 & 1 & 1 & -2 & 4 \end{array}\right]$$

$$\begin{array}{c} \sim \\ -\text{R1} \\ \\ \\ \end{array} \left[\begin{array}{rrr|rr} 1 & 1 & 1 & -2 & -1 \\ 1 & -1 & -1 & 3 & 2 \\ 1 & 1 & -1 & 4 & 3 \\ 1 & 1 & 1 & -2 & 4 \end{array}\right]$$

$$\begin{array}{c} \sim \\ -\text{R1}+\text{R2} \\ -\text{R1}+\text{R3} \\ -\text{R1}+\text{R4} \end{array} \left[\begin{array}{rrr|rr} 1 & 1 & 1 & -2 & -1 \\ 0 & -2 & -2 & 5 & 3 \\ 0 & 0 & -2 & 6 & 4 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right].$$

At this point, we see that if we use the last solution column, the system will be inconsistent. Thus, $\underline{u}$ is not in the span of *S*. We can now examine the augmented matrix without this solution column:

$$\left[\begin{array}{rrr|r} 1 & 1 & 1 & -2 \\ 0 & -2 & -2 & 5 \\ 0 & 0 & -2 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{c} \sim \\ -\frac{1}{2}\text{R2} \\ -\frac{1}{2}\text{R3} \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 1 & -2 \\ 0 & 1 & 1 & -\frac{5}{2} \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

This system is consistent, so $\underline{v}$ is in the span of *S*.

4. Which of the following sets of vectors are linearly independent?

(a) $\{(0,1,-1),(-1,0,1),(1,-1,0)\}$

Solution:

We create a matrix whose rows are the vectors in this set and reduce it:

$$\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\underset{\text{R1} \longleftrightarrow \text{R3}}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\underset{\text{R1}+\text{R2}}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\underset{-\text{R2}}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}$$

$$\underset{-\text{R2}+\text{R3}}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

There is a row of zeros, so the set is linearly dependent.

(b) $\{(-1,1,1,1),(-1,-1,1,1),(-1,-1,-1,1)\}$

Solution:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$\underset{-\mathsf{R1}}{\sim} \quad \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$\underset{\substack{\mathsf{R1}+\mathsf{R2} \\ \mathsf{R1}+\mathsf{R3}}}{\sim} \quad \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & -2 & 0 \end{bmatrix}$$

$$\underset{\substack{-\frac{1}{2}\mathsf{R2} \\ -\frac{1}{2}\mathsf{R3}}}{\sim} \quad \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\underset{\substack{\mathsf{R2}+\mathsf{R1} \\ -\mathsf{R2}+\mathsf{R3}}}{\sim} \quad \begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\underset{\mathsf{R3}+\mathsf{R1}}{\sim} \quad \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

This has been reduced as far as it can go, and it has no row of zeros. Consequently, this set is linearly independent.

5. Select a linearly independent subset of

$$S = \{(1,1,1,1),\ (0,2,-1,0),\ (1,3,0,1),\ (3,3,1,3)\}$$

that spans the same subspace of $\mathbb{R}^4$ as S does.

<u>**Solution**</u>**:**

There are two ways in which this may be done.

Method 1: We can create a matrix whose rows are the vectors in *S* and use any elementary operation *except* switching two rows:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 1 & 3 & 0 & 1 \\ 3 & 3 & 1 & 3 \end{bmatrix}$$

$$\underset{\substack{-\mathrm{R1}+\mathrm{R3} \\ -3\mathrm{R1}+\mathrm{R4}}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\underset{-\mathrm{R2}+\mathrm{R3}}{\sim} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\underset{\substack{\frac{1}{2}\mathrm{R2} \\ -\frac{1}{2}\mathrm{R4}}}{\sim} \begin{bmatrix} \textcircled{1} & 1 & 1 & 1 \\ 0 & \textcircled{1} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & 0 \end{bmatrix}.$$

There are leading ones in rows 1, 2, and 4, so we select the vectors we had originally put in rows 1, 2, and 4. In other words, we select the subset

$$\{(1,1,1,1),\ (0,2,-1,0),\ (3,3,1,3)\}.$$

Method 2: We create a matrix whose columns are the vectors in S and use any elementary operation:

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ 1 & 2 & 3 & 3 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{matrix} \sim \\ -\mathsf{R1}+\mathsf{R2} \\ -\mathsf{R1}+\mathsf{R3} \\ -\mathsf{R1}+\mathsf{R4} \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 2 & 2 & 0 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \sim \\ \frac{1}{2}\mathsf{R2} \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \sim \\ \mathsf{R2}+\mathsf{R3} \end{matrix} \quad \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} \sim \\ -\frac{1}{2}\mathsf{R3} \end{matrix} \quad \begin{bmatrix} \textcircled{1} & 0 & 1 & 3 \\ 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & 0 & \textcircled{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are leading ones in columns 1, 2, and 4, so we select the vectors we had originally put in columns 1, 2, and 4. In other words, we select the subset

$$\{(1,1,1,1),\ (0,2,-1,0),\ (3,3,1,3)\}.$$

6. Find a basis for the span of the set of vectors

$$\{(-1,1,1,1),\ (0,1,-1,1),\ (1,0,-2,0),\ (1,1,-1,0)\}.$$

Solution:

We create a matrix whose rows are the vectors in the set and reduce:

$$\begin{bmatrix} -1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & -2 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\underset{-\text{R1}}{\sim} \quad \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & -2 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$$

$$\underset{\substack{-\text{R1}+\text{R3} \\ -\text{R1}+\text{R4}}}{\sim} \quad \begin{bmatrix} 1 & -1 & -1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$$\underset{\substack{-\text{R1}+\text{R3} \\ -\text{R1}+\text{R4}}}{\sim} \quad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 \end{bmatrix}$$

$$\underset{\frac{1}{2}\text{R4}}{\sim} \quad \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

$$\underset{\substack{R4+R2 \\ 2R4+R1}}{\sim} \begin{bmatrix} \textcircled{1} & 0 & 0 & -1 \\ 0 & \textcircled{1} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \textcircled{1} & -\frac{1}{2} \end{bmatrix}.$$

Now, we can select the nonzero rows from this matrix to form a basis:

$$\left\{(1,0,0,-1),\ \left(0,1,0,\frac{1}{2}\right),\ \left(0,0,1,-\frac{1}{2}\right)\right\}.$$

Alternately, there are leading ones in rows 1, 2, and 4. Since we did not switch any rows, we can select the vectors we had originally put in rows 1, 2, and 4 to form a basis:

$$\{(-1,1,1,1),\ (0,1,-1,1),\ (1,1,-1,0)\}.$$

7. Determine if the set of vectors $\{(-1,1,1),\ (-1,1,-1),\ (-1,-1,1)\}$ is a basis for $\mathbb{R}^3$.

<u>Solution</u>:

As usual, we form a matrix whose rows are the vectors in the set and reduce:

$$\begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\underset{-R1}{\sim} \begin{bmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\underset{\substack{R1+R2\\R1+R3}}{\sim}\begin{bmatrix}1&-1&-1\\0&0&-2\\0&-2&0\end{bmatrix}$$

$$\underset{R2\longleftrightarrow R3}{\sim}\begin{bmatrix}1&-1&-1\\0&-2&0\\0&0&-2\end{bmatrix}$$

$$\underset{\substack{-\frac{1}{2}R2\\-\frac{1}{2}R3}}{\sim}\begin{bmatrix}1&-1&-1\\0&1&0\\0&0&1\end{bmatrix}$$

$$\underset{R2+R1}{\sim}\begin{bmatrix}1&0&-1\\0&1&0\\0&0&1\end{bmatrix}$$

$$\underset{R3+R1}{\sim}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$

There is no row of zeros, so the set is linearly independent. Also, the span of the set is the row space of the above matrix, which is $\mathbb{R}^3$. Therefore, the set *is* indeed a basis for $\mathbb{R}^3$.

8. Let $A=\begin{bmatrix}1&1&0&1&1\\-1&1&1&0&1\\0&2&1&1&2\end{bmatrix}$. Find the dimension of:

(a) the row space of A,

Solution:

We begin by reducing A: $\begin{bmatrix}1&1&0&1&1\\-1&1&1&0&1\\0&2&1&1&2\end{bmatrix}$

$$\underset{R1+R2}{\sim} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 & 2 \end{bmatrix}$$

$$\underset{-R2+R3}{\sim} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\frac{1}{2}R2}{\sim} \begin{bmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\underset{-R2+R1}{\sim} \begin{bmatrix} \textcircled{1} & 0 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \textcircled{1} & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

There are two nonzero rows (or two leading ones), so any basis of the row space of A contains two vectors. Consequently, the dimension of the row space of A is 2.

(b) the solution set of the equation $A\underline{x} = 0$ (note that $\underline{x}$ is a column vector).

Solution:

The augmented matrix for this system is

$$\left[\begin{array}{rrrrr|r} 1 & 1 & 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 & 2 & 0 \end{array}\right].$$

From the work done in part (a), the reduced echelon form of the augmented matrix is

$$\left[\begin{array}{ccccc|c} 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Using the variables v, w, x, y, and z, the solution to this system is

$$\left\{\begin{array}{rcl} v & = & \frac{1}{2}x - \frac{1}{2}y \\ w & = & -\frac{1}{2}x - \frac{1}{2}y - z \\ x & = & x \\ y & = & y \\ z & = & z. \end{array}\right.$$

Then, every solution has the form

$$\begin{bmatrix} v \\ w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x - \frac{1}{2}y \\ -\frac{1}{2}x - \frac{1}{2}y - z \\ x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

So, the set $\left\{\left(\frac{1}{2}, -\frac{1}{2}, 1, 0, 0\right),\ \left(-\frac{1}{2}, -\frac{1}{2}, 0, 1, 0\right),\ (0, -1, 0, 0, 1)\right\}$ is a basis of the solution set of this system. Ergo, the dimension of the solution set of $A\underline{x} = 0$ is 3.