NEW FAMILIES OF STRONGLY REGULAR GRAPHS

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Abstract. We apply symmetric balanced generalized weighing matrices with zero diagonal to construct four parametrically new infinite families of strongly regular graphs.

1. Introduction

Let G be a multiplicatively written finite group. A matrix $W = [\alpha_{ij}]$ of order v with entries from the set $\overline{G} = G \cup \{0\}$ is called a *balanced generalized weighing matrix with parameters* (v, k, λ), or a $BGW(v, k, \lambda)$, over G if each row of W contains exactly k nonzero entries and, for all distinct $i, h \in \{1, 2, \ldots, v\}$, the multiset $\{\alpha_{hj}^{-1}\alpha_{ij} : 1 \le j \le v, \alpha_{ij} \ne 0, \alpha_{hj} \ne 0\}$ contains exactly $\lambda/|G|$ copies of each element of G.

If W is a $BGW(v, k, \lambda)$ over a group G, then so is W^T. If f is a homomorphism from G onto a group G', then replacing every nonzero entry of W by its image under f yields a $BGW(v, k, \lambda)$ over G'.

Most of the known balanced generalized weighing matrices belong to the family

$$BGW\left(\frac{q^{d+1}-1}{q-1}, q^d, q^d - q^{d-1}\right) \text{ over } G, \tag{1}$$

where q is a prime power, d is a positive integer, and G is a cyclic group whose order divides $q - 1$. Since such a group G can be regarded as a subgroup of the multiplicative group of the field $GF(q)$, matrix (1) can be regarded as a matrix over $GF(q)$.

Let ω be a generator of a finite cyclic group G. A matrix $W = [\alpha_{ij}]$ of order v over $\overline{G}$ is called *ω-circulant* if, for $i, j = 2, 3, \ldots, v$, $\alpha_{ij} = \alpha_{i-1,j-1}$ and $\alpha_{i1} = \omega\alpha_{i-1,v}$. As Jungnickel showed in [6], there is always an ω-circulant balanced generalized weighing matrix with parameters (1).

Another useful representation of balanced generalized weighing matrices is considering them as matrices over a group ring. Let R be the group ring of a finite group G over the rationals. For any $x \in \overline{G}$, let $x^* = x^{-1}$ if $x \in G$ and $x^* = 0$ if $x = 0$. For a matrix $W = [\alpha_{ij}]$ of order v with entries from $\overline{G}$, let $W^* = [\alpha_{ij}^*]^T$. Then W is a $BGW(v, k, \lambda)$ over G if and only if the following equation over R is satisfied:

$$WW^* = \left(ke - \frac{\lambda}{|G|}G\right)I + \left(\frac{\lambda}{|G|}G\right)J, \tag{2}$$

where e is the identity element of G, G stands for the sum of all elements of G in R, I is the identity matrix of order v, and J is the matrix of order v with all entries equal to e.

For further details on balanced generalized weighing matrices, see [6] and [4].

A *strongly regular graph with parameters* (v, k, λ, μ), or an $SRG(v, k, \lambda, \mu)$, is a simple graph Γ with v vertices, not complete or null, in which the number of common neighbors of vertices x and

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y is k, λ, or μ according as x and y are equal, adjacent, or non-adjacent, respectively. If A is an adjacency matrix of the graph Γ, then

$$A^2 = (k-\mu)I + (\lambda-\mu)A + \mu J \tag{3}$$

and

$$AJ = JA = kJ. \tag{4}$$

Conversely, if an adjacency matrix A of a simple graph Γ (not complete or null) satisfies (3) and (4), then Γ is strongly regular.

A *symmetric* (v,k,λ)-*design* is a pair $\mathbf{D} = (X, \mathcal{B})$, where X is a set (of *points*) of cardinality v and $\mathcal{B}$ is a set of v subsets of X (*blocks*), each of cardinality k, such that any 2-subset of X is contained in exactly λ blocks. If $X = \{x_1, x_2, \ldots, x_v\}$ and $\mathcal{B} = \{B_1, B_2, \ldots, B_v\}$, then the design $\mathbf{D}$ can be described by its *incidence matrix*, that is a $(0,1)$ matrix of order v whose (i,j)-entry is equal to 1 if and only if $x_i \in B_j$. A $(0,1)$ matrix N of order v is an incidence matrix of a symmetric (v,k,λ)-design if and only if

$$NN^T = (k-\lambda)I + \lambda J.$$

An incidence matrix N of a symmetric (v,k,λ)-design also satisfies the equation $NJ = JN = kJ$. Therefore, if it is a symmetric matrix with all diagonal entries equal to 0, then N is an adjacency matrix of an $SRG(v,k,\lambda,\lambda)$. For further information on strongly regular graphs and symmetric designs see [1], [2], [3], and [8].

In this paper we construct four infinite families of symmetric designs that admit a symmetric incidence matrix with zero diagonal and thus obtain four infinite families of strongly regular graphs. Symmetric designs with these parameters were constructed in [5] but the parameters of the strongly regular graphs are new.

2. Symmetric balanced generalized weighing matrices with zero diagonal

In this section we describe a construction of an infinite family of symmetric balanced generalized weighing matrices with zero diagonal. It is a modification of the construction described in [7]

Let $\mathcal{M}$ be a nonempty set of $m \times n$ matrices and let S be a group of bijections $\mathcal{M} \to \mathcal{M}$. If $W = [\alpha_{ij}]$ is a $BGW(w,l,\mu)$ over S, then, for $X \in \mathcal{M}$, we denote by $W \otimes X$ the block matrix $[\alpha_{ij}X]$, where, for $\alpha_{ij} \in S$, $\alpha_{ij}X$ is the image of X under the bijection α_{ij}; if $\alpha_{ij} = 0$, then $\alpha_{ij}X$ is the $m \times n$ zero matrix.

Theorem 2.1. *Let $v \geq k \geq \lambda$ be positive integers and let G be a finite group. Let $\mathcal{M}$ be a nonempty set of matrices, each of which is a $BGW(v,k,\lambda)$ over G, and let S be a group of bijections $\mathcal{M} \to \mathcal{M}$ such that $(\sigma X)(\sigma Y)^* = XY^*$ for all $\sigma \in S$ and all $X, Y \in \mathcal{M}$. Suppose there exists a balanced generalized weighing matrix W over S with parameters (w,l,μ) such that*

$$\sum_{\sigma \in S} \sigma X = \left(\frac{\lambda l |S|}{k\mu|G|} G\right) J \tag{5}$$

for all $X \in \mathcal{M}$. Then, for each $X \in \mathcal{M}$, the matrix $W \otimes X$ is a $BGW(vw, kl, \lambda l)$ over G.

Proof. Let $W = [\alpha_{ij}]$, $i,j = 1,2,\ldots,w$. If $i \in \{1,2,\ldots,w\}$, then there exists $\sigma_1, \sigma_2, \ldots, \sigma_l \in S$ such that

$$\sum_{j=1}^{w} (\alpha_{ij}X)(\alpha_{ij}X)^* = \sum_{j=1}^{l} (\sigma_j X)(\sigma_j X)^* = lXX^* = \left(kle - \frac{\lambda l}{|G|}G\right) I + \left(\frac{\lambda l}{|G|}G\right) J.$$

If $i, h \in \{1, 2, \ldots, w\}$ and $i \neq h$, then there exist $\sigma_1, \sigma_2, \ldots, \sigma_\mu \in S$ and $\tau_1, \tau_2, \ldots, \tau_\mu \in S$ such that

$$\sum_{j=1}^{w} (\alpha_{ij}X)(\alpha_{hj}X)^* = \sum_{j=1}^{\mu} (\sigma_j X)(\tau_j X)^* = \sum_{j=1}^{\mu} (\tau_j^{-1}\sigma_j X)X^*$$
$$= \frac{\mu}{|S|} \sum_{\sigma \in S} (\sigma X)X^* = \left(\frac{\lambda l}{k|G|} G\right) JX^* = \frac{\lambda l}{k|G|} J(GX^*) = \left(\frac{\lambda l}{|G|} G\right) J.$$

Thus, $W \otimes X$ satisfies (2) and therefore it is a $BGW(vw, kl, \lambda l)$ over G. □

We now describe a possible realization of the conditions of Theorem 2.1.

Let q be a prime power, d a positive integer, n a divisor of $q-1$, and $G = \prec \omega \succ$ a cyclic group of order n. Let $\mathcal{M}$ be the set of all $BGW(v, q^d, q^d - q^{d-1})$ matrices over G with $v = (q^{d+1}-1)/(q-1)$. For the remainder of this section, $\rho\colon \mathcal{M} \to \mathcal{M}$ is the map defined as follows: if $X = [x_{ij}] \in \mathcal{M}$, then $\rho X = [x'_{ij}]$ with

$$x'_{ij} = \begin{cases} x_{i,j-1} & \text{if } 2 \le j \le v, \\ \omega x_{iv} & \text{if } j = 1. \end{cases}$$

Let S be the cyclic group generated by ρ. Then $|S| = nv$. Let $r = q^{d+1}$. Then, for any positive integer m, there exists a $BGW(w, r^m, r^m - r^{m-1})$ over S with $w = (r^{m+1}-1)/(r-1)$. Let W be such a matrix.

Theorem 2.2. *For any $X \in \mathcal{M}$, the matrix $W \otimes X$ is a $BGW(vw, q^d r^m, (q^d - q^{d-1})r^m)$ over G.*

Proof. Let $X, Y \in \mathcal{M}$, $X = [x_{ij}]$, $Y = [y_{ij}]$. The (i,j)-entry of XY^* is equal to $\sum_{t=1}^{v} x_{it}y^*_{jt}$ and the (i,j)-entry of $(\rho X)(\rho Y)^*$ is equal to $\sum_{t=1}^{v-1} x_{it}y^*_{jt} + (\omega x_{iv})(\omega x_{jv})^*$, so $(\rho X)(\rho Y)^* = XY^*$. The (i,j)-entry of $\sum_{\sigma \in S} \sigma X$ is

$$\sum_{s=1}^{n} \sum_{t=1}^{v} \omega^s x_{it} = \sum_{t=1}^{v} G x_{it} = q^d G,$$

so all the conditions of Theorem 2.1 are satisfied. □

The next theorem is a slight generalization of a result obtained in [7]. It is crucial for subsequent constructions.

Theorem 2.3. *Let q be a prime power, n a divisor of $q-1$, and G a cyclic group of order n. If $q(q-1)/n$ is even, then, for any positive integer d, there exists a symmetric balanced generalized weighing matrix*

$$BGW\left(\frac{q^{2d}-1}{q-1}, q^{2d-1}, q^{2d-1} - q^{2d-2}\right) \tag{6}$$

over G with all diagonal entries equal to 0.

Proof. We begin with the case $d = 1$. Let $GF(q) = \{a_1, a_2, \ldots, a_q\}$. Define the matrix $U = [\alpha_{ij}]$ of order $q+1$ by

$$\alpha_{ij} = \begin{cases} a_i - a_j & \text{if } 1 \le i \le q, 1 \le j \le q, \\ 1 & \text{if } 1 \le i \le q, j = q+1, \\ -1 & \text{if } i = q+1, 1 \le j \le q, \\ 0 & \text{if } i = j = q+1. \end{cases}$$

It is well known and readily verified that U is a $BGW(q+1, q, q-1)$ over $GF(q)^*$. Let $V = [\alpha_{ij}^{(q-1)/n}]$. If q is odd, then $(q-1)/n$ is even, and therefore V is a symmetric $BGW(q+1, q, q-1)$

with zero diagonal over a cyclic group of order n. It is true for q even too, since in this case $GF(q)$ is a field of characteristic 2.

We now consider the general case and let $v = (q^{d+1} - 1)/(q - 1)$. Let ω be a generator of G and let $\mathcal{M}_c$ be the set of all ω-circulant $BGW(v, q^d, q^d - q^{d-1})$ matrices over G. Let R be the matrix of order v with all back diagonal entries equal to 1 and all other entries equal to 0. Then, for any $X \in \mathcal{M}_c$, XR is a symmetric $BGW(v, q^d, q^d - q^{d-1})$ over G. Let $\mathcal{M}_s = \{XR\colon X \in \mathcal{M}_c\}$. Then $\rho Y \in \mathcal{M}_s$ for all $Y \in \mathcal{M}_s$, so the cyclic group S of order nv generated by ρ can be regarded as a group of bijections on $\mathcal{M}_s$. Since nv divides $q^{d+1} - 1$ and $q^{d+1}(q^{d+1} - 1)/(nv)$ is even, there exists a symmetric $BGW(q^{d+1} + 1, q^{d+1}, q^{d+1} - 1)$ over S with zero diagonal. Let W be such a matrix and let $Y \in \mathcal{M}_s$. By Theorem 2.2, $W \otimes Y$ is a BGW with parameters (6) over G. Since both Y and W are symmetric, so is $W \otimes Y$. Since the diagonal entries of W are zeros, so are the diagonal entries of $W \otimes Y$. $\square$

3. Symmetric designs with symmetric incidence matrices

Theorem 2.1 applied to the trivial group G yields the following result first obtained in [5].

Theorem 3.1. *Suppose that a nonempty set $\mathcal{M}$ of incidence matrices of symmetric (v, k, λ)-designs and a finite group S of bijections $\mathcal{M} \to \mathcal{M}$ satisfy conditions*

(i) *$(\sigma M)(\sigma N)^T = MN^T$ for all $M, N \in \mathcal{M}$ and all $\sigma \in S$ and*

(ii) *for each $M \in \mathcal{M}$, the matrix $\sum_{\sigma \in S} \sigma M$ is constant.*

If W is a $BGW(w, l, \mu)$ over S with $k^2\mu = v\lambda l$, then, for $N \in \mathcal{M}$, $W \otimes N$ is an incidence matrix of a symmetric $(vw, kl, \lambda l)$-design.

Remark 3.2. Since each matrix $M \in \mathcal{M}$ has constant row sum k, condition (ii) of Theorem 3.1 implies that $\sum_{\sigma \in S} \sigma M = (k|S|)/v)J$, in agreement with the corresponding condition imposed by Theorem 2.1.

Using Theorem 2.1, the first author constructed in [5] four infinite families of symmetric designs, where the starting symmetric design is the development of a McFarland difference set or its complement or a Spence difference set or its complement.

In order to obtain symmetric designs whose incidence matrices can serve as adjacency matrices of strongly regular graphs, we will need the starting symmetric designs with symmetric incidence matrices. In this section we develop the tools necessary for obtaining such matrices.

We begin with introducing a certain order on a finite abelian group.

Lemma 3.3. *Let G be a finite abelian group. It is possible to order elements of $G = \{x_1, x_2, \ldots, x_n\}$ so that $x_i + x_{n+1-i}$ is the same for $i = 1, 2, \ldots, n$.*

Proof. For each $a \in G$, let $H(a) = \{x \in G\colon 2x = a\}$. Since the sets $H(a)$ are pairwise disjoint, either all of them are singletons or at least one of them is empty. Fix $a \in G$ such that $|H(a)| \le 1$ and partition the set $G \setminus H(a)$ into 2-subsets $\{b_i, c_i\}$ such that $b_i + c_i = a$. For $1 \le i \le \frac{n}{2}$, put $x_i = b_i$ and $x_{n+1-i} = c_i$. If $H(a) \ne \varnothing$, then n is odd, and we let $x_{(n+1)/2}$ be the element of $H(a)$. $\square$

We will call the order on G described in Lemma 3.3 *symmetric*. From now on, we will always assume that a finite abelian group G is equipped with a symmetric order, and $G = \{x_1, x_2, \ldots, x_n\}$ means that $x_i + x_{n+1-i}$ is the same for $i = 1, 2, \ldots, n$.

With any subset A of a finite abelian group $G = \{x_1, x_2, \ldots, x_n\}$ we associate a $(0, 1)$ matrix $M(A) = [m_{ij}(A)]$ of order n, where

$$m_{ij}(A) = \begin{cases} 1 & \text{if } x_{n+1-i} - x_j \in A, \\ 0 & \text{if } x_{n+1-i} - x_j \notin A. \end{cases}$$

The definition of symmetric order implies that matrices $M(A)$ are symmetric. Importance of this fact is demonstrated by the following theorem.

Theorem 3.4. *Let $v > k > \lambda$ be positive integers and let $\mathcal{M}$ me a non-empty set of symmetric incidence matrices of symmetric (v, k, λ)-designs. Let S be a group of bijections $\mathcal{M} \to \mathcal{M}$ that satisfies conditions* (i) *and* (ii) *of Theorem 3.1. Let W be a symmetric $BGW(w, l, \mu)$ over S with zero diagonal. If $k^2\mu = v\lambda l$, then, for $N \in \mathcal{M}$, $W \otimes N$ is an adjacency matrix of a strongly regular graph with parameters $(vw, kl, \lambda l, \lambda l)$.*

Proof. Let $W = [\alpha_{ij}]$, $i, j = 1, 2, \ldots, w$. By Theorem 3.1, $W \otimes N$ is an incidence matrix of a symmetric $(vw, kl, \lambda l)$-design. Since each matrix $\alpha_{ij}N$ is symmetric and $\alpha_{ij} = \alpha_{ji}$ for $i, j = 1, 2, \ldots, w$, $W \otimes N$ is a symmetric matrix. Since $\alpha_{ii} = 0$ for $i = 1, 2, \ldots, w$, the diagonal entries of $W \otimes N$ are equal to 0. Therefore, $W \otimes N$ is an adjacency matrix of a strongly regular graph with parameters $(vw, kl, \lambda l, \lambda l)$. □

Remark 3.5. If parameters (w, l, μ) of the matrix W in Theorem 3.4 are those given by Theorem 2.3, then the condition $k^2\mu = v\lambda l$ is equivalent to $q = k^2/(k - \lambda)$.

We conclude this section with the following technical lemma.

Lemma 3.6. *If A and B are subsets of a finite abelian group $G = \{x_1, x_2, \ldots, x_n\}$, then* (i) *for $i, j = 1, 2, \ldots, n$, the (i, j)-entry of the matrix $M(A)M(B)^T$ is equal to $|(A + x_i) \cap (B + x_j)|$ and* (ii) *$M(A)J = |A|J$.*

Proof. The (i, j)-entry of $M(A)M(B)^T$ is equal to the number of indices k such that $x_{n+1-k} - x_i \in A$ and $x_{n+1-k} - x_j \in B$. This implies (i). The statement (ii) is immediate. □

4. McFarland and Spence designs

In this section, we construct designs with parameters of McFarland and Spence difference sets, which have symmetric incidence matrices.

Let q be a prime power, d a positive integer, and V the $(d+1)$-dimensional vector space over the field $GF(q)$. We will call subspaces of V of dimension d *hyperplanes* and their cosets *d-flats.* If H is a hyperplane and $x, y \in V$, we will call d-flats $H + x$ and $H + y$ *parallel.* The space V contains $r = (q^{d+1} - 1)/(q - 1)$ hyperplanes, which we denote $H_1, H_2, \ldots, H_r$. All d-flats parallel to H_i form a *parallel class* Π_i, $|\Pi_i| = q$. For $i = 1, 2, \ldots, r$, we fix a cyclic permutation π_i of the parallel class Π_i.

Let $\mathcal{F}$ be the set consisting of all d-flats, their complements, the empty set, and the entire space V (so $|\mathcal{F}| = 2(qr + 1)$). We define a bijection $\pi\colon \mathcal{F} \to \mathcal{F}$ as follows: if $F \in \Pi_i$, then $\pi(F) = \pi_i(F)$ and $\pi(V \setminus F) = V \setminus \pi_i(F)$; $\pi(\varnothing) = \varnothing$; $\pi(V) = V$.

We will regard V as an abelian group equipped by a symmetric order, so, for any subset A of V, a symmetric matrix $M(A)$ is defined .

Lemma 4.1. *For $A, B \in \mathcal{F}$,* (i) *$M(\pi A)M(\pi B)^T = M(A)M(B)^T$ and* (ii) *$\sum_{k=1}^{q} M(\pi^k A)$ is a constant matrix.*

Proof. Statement (i) follows immediately from Lemma 3.6. To prove (ii) note that if A is a d-flat, then the d-flats $\pi^k A$, $k = 1, 2, \ldots, q$, partition V. Therefore, $\sum_{j=1}^{q} M(\pi^k A) = J$ and $\sum_{j=1}^{q} M(\pi^k(V \setminus A)) = (q - 1)J$. Of course, $\sum_{j=1}^{q} M(\pi^k(\varnothing)) = O$ and $\sum_{j=1}^{q} M(\pi^k V) = qJ$. □

Let n be a positive integer and let $\mathcal{F}^n$ be the set of all ordered n-tuples of elements of $\mathcal{F}$. For $\mathbf{A} = (A_1, A_2, \ldots, A_n) \in \mathcal{F}^n$, we define a symmetric $(0, 1)$ matrix $N(\mathbf{A})$ of order nq^{d+1} as a block matrix $[N_{ij}(\mathbf{A})]$, $i, j = 1, 2, \ldots, n$, where

$$N_{ij}(\mathbf{A}) = \begin{cases} M(A_{i+j-1}) & \text{if } i+j \le n+1, \\ M(\pi(A_{i+j-n-1})) & \text{if } i+j \ge n+2. \end{cases}.$$

If $\mathbf{A} = (A_1, A_2, \ldots, A_n) \in \mathcal{F}^n$, we denote by $\overline{\mathbf{A}}$ the complementary n-tuple $(V \setminus A_1, V \setminus A_2, \ldots, V \setminus A_n)$. Clearly, $N(\overline{\mathbf{A}}) = J - N(\mathbf{A})$.

We will now introduce McFarland and Spence symmetric designs with symmetric incidence matrices. Recall that $r = (q^{d+1} - 1)/(q - 1)$ is the number of hyperplanes in V.

Definition 4.2. A *McFarland* $(r+1)$*-tuple* is an $(r+1)$-tuple $\mathbf{A} = (A_1, A_2, \ldots, A_{r+1}) \in \mathcal{F}^{r+1}$ such that one of the sets $A_1, A_2, \ldots, A_{r+1}$ is empty and the other r are pairwise non-parallel d-flats. A *Spence* r*-tuple* is an r-tuple $\mathbf{A} = (A_1, A_2, \ldots, A_r) \in \mathcal{F}^r$ such that one of the sets $A_1, A_2, \ldots, A_r$ is the complement of a d-flat parallel to H_1 and the other r are pairwise non-parallel d-flats, which are not parallel to H_1.

Theorem 4.3. *Let $[A_{ij}]$, $i, j = 1, 2, \ldots, r+1$, be an array of subsets of V, all rows and all columns of which are McFarland $(r+1)$-tuples. Then the block matrix $N = [M(A_{ij})]$ is an incidence matrix of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design.*

Proof. For $i, h = 1, 2, \ldots, r+1$, let $S_{ih} = \sum_{j=1}^{r+1} M(A_{ij}) M(A_{hj})^T$. By Lemma 3.6, for $k, l = 1, 2, \ldots, q^{d+1}$, the (k, l)-entry of S_{ih} is equal to $\sum_{j=1}^{r+1} |(A_{ij} + x_k) \cap (A_{hj} + x_l)|$.

Let $i = h$. If A_{ij} is a d-flat, then let it be parallel to a hyperplane H. In this case, $A_{ij} + x_k = A_{ij} + x_l$ or $(A_{ij} + x_k) \cap (A_{hj} + x_l) = \varnothing$ depending on whether $x_k - x_l$ is or is not in H. Therefore, the (k, l)-entry of S_{ih} is equal to rq^d if $k = l$, and it is equal to $q^d(q^d - 1)/(q - 1) = (r-1)q^{d-1}$ if $k \ne l$.

Let $i \ne h$. Then either $A_{ij} + x_k$ and $A_{hj} + x_l$ are non-parallel d-flats, which meet in q^{d-1} points, or one of these sets is empty. Therefore, the (k, l)-entry of S_{ih} is equal to $(r-1)q^{d-1}$. The proof is now complete. □

Corollary 4.4. *If $\mathbf{A}$ is a McFarland $(r+1)$-tuple, then $N(\mathbf{A})$ is a symmetric incidence matrix of a symmetric $((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$-design.*

Symmetric designs constructed in Theorem 4.3 have parameters of McFarland difference sets. Symmetric designs with parameters of Spence difference sets can be obtained in a similar manner.

Theorem 4.5. *Let V be the $(d+1)$-dimensional vector space over $GF(3)$ and let $r = (3^{d+1} - 1)/2$. Let $[A_{ij}]$, $i, j = 1, 2, \ldots, r$, be an array of subsets of V, all rows and all columns of which are Spence r-tuples. Then the block matrix $N = [M(A_{ij})]$ is an incidence matrix of a symmetric $(3^{d+1}(3^{d+1} - 1)/2, 3^d(3^{d+1} + 1)/2, 3^d(3^d + 1)/2)$-design.*

Corollary 4.6. *If $\mathbf{A}$ is a Spence r-tuple over $GF(3)$, then $N(\mathbf{A})$ is a symmetric incidence matrix of a symmetric $(3^{d+1}(3^{d+1} - 1)/2, 3^d(3^{d+1} + 1)/2, 3^d(3^d + 1)/2)$-design.*

5. Constructing strongly regular graphs

In this section we apply Theorem 3.4 to the set of matrices $\{N(\mathbf{A}) \colon \mathbf{A} \in \mathcal{M}\}$, where $\mathcal{M}$ is the set of McFarland $(r+1)$-tuples or the set of their complements or the set of Spence r-tuples or the set of their complements.

We begin with defining bijections $\pi, \sigma \colon \mathcal{F}^n \to \mathcal{F}^n$ as follows:

$$\pi(A_1, A_2, \ldots, A_n) = (\pi A_1, \pi A_2, \ldots, \pi A_n);$$

$$\sigma(A_1, A_2, \ldots, A_n) = (A_2, A_3, \ldots, A_n, A_1).$$

Observe that $\pi\sigma = \sigma\pi$ and that π and σ generate cyclic groups of orders q and n, respectively. If $n = r = (q^{d+1} - 1)/(q - 1)$, then q and n are relatively prime; if $n = r + 1$ and q is odd, then again q and n are relatively prime. From now on, we assume that q is an odd prime power, $n = r$ or $r + 1$, and G is the cyclic group of order nq generated by π and σ.

Lemma 5.1. *For $\mathbf{A}, \mathbf{B} \in \mathcal{F}^n$ and $\alpha \in G$,*
(i) $N(\alpha\mathbf{A})N(\alpha\mathbf{B})^\top = N(\mathbf{A})N(\mathbf{B})^\top$*,*
(ii) $\sum_{k=1}^{q}\sum_{l=1}^{n} N(\pi^k\sigma^l\mathbf{A})$ *is a constant matrix.*

Proof. (i) is straightforward for $\alpha = \pi$ and $\alpha = \sigma$ and therefore it is true for any $\alpha \in G$.

If $\mathbf{A} = (A_1, A_2, \ldots, A_n)$, then

$$\sum_{k=1}^{q}\sum_{l=1}^{n} N(\pi^k\sigma^l\mathbf{A}) = [S_{ij}],$$

where each block S_{ij} is equal to $\sum_{j=1}^{n}\sum_{k=1}^{q} M(\pi^k A_j)$, and we apply Lemma 4.1(ii). $\square$

Let $\mathcal{P}$ be the set of all matrices $N(\mathbf{A})$, where $\mathbf{A}$ is a McFarland $(r+1)$-tuple. For $P = N(\mathbf{A}) \in \mathcal{P}$ and $\alpha \in G$, let $\alpha P = N(\alpha\mathbf{A})$. The set $\mathcal{P}$ and the cyclic group G of bijections $\mathcal{P} \to \mathcal{P}$ satisfy conditions (i) and (ii) of Theorem 3.4. If $(v, k, \lambda) = ((r+1)q^{d+1}, rq^d, (r-1)q^{d-1})$, then $k^2/(k-\lambda) = r^2$. We have $r^2(r^2-1)/|S| = r^2(r-1)/q$. Therefore, if r is a prime power and $r(r-1)/q$ is even, Theorem 2.3 yields, for any positive integer m, a symmetric $BGW((r^{4m}-1)/(r^2-1), r^{4m-2}, r^{4m-2} - r^{4m-4})$ with zero diagonal. This leads to the following result.

Theorem 5.2. *Let q be an odd prime power and d a positive integer. If $r = (q^{d+1} - 1)/(q - 1)$ is a prime power, then, for any positive integer m, there exists a strongly regular graph with parameters*

$$\left(\frac{q^{d+1}(r^{4m}-1)}{r-1}, q^d r^{4m-1}, q^{d-1}r^{4m-2}(r-1), q^{d-1}r^{4m-2}(r-1)\right).$$

Let $\mathcal{P}$ be the set of all matrices $N(\mathbf{A})$, where $\mathbf{A}$ is the complement of a McFarland $(r+1)$-tuple. Then again any $\alpha \in G$ can be regarded as a bijection $\mathcal{P} \to \mathcal{P}$. The complement of a McFarland symmetric design has parameters $((r+1)q^{d+1}, q^d(q^{d+1}+q-1), q^d(q^d+1)(q-1))$. In this case we want $s = q^{d+1} + q - 1$ to be a prime power and $s^2(q^d+1)(q-1)$ to be even.

Theorem 5.3. *Let q is an odd prime power and d a positive integer. If $s = q^{d+1} + q - 1$ is a prime power, then, for any positive integer m, there exists a strongly regular graph with parameters*

$$\left(\frac{q^{d+1}(s^{4m}-1)}{(q-1)(s+1)}, q^d s^{4m-1}, q^d s^{4m-2}(q^d+1)(q-1), q^d s^{4m-2}(q^d+1)(q-1)\right).$$

Let $\mathcal{P}$ be the set of all matrices $N(\mathbf{A})$, where $\mathbf{A}$ is a Spence r-tuple over $GF(3)$ with $r = (3^{d+1} - 1)/2$. In this case, $|S| = 3r$, and we want $q = (3^{d+1} + 1)/2$ to be a prime power.

Theorem 5.4. *Let d be a positive integer. If $q = (3^{d+1} + 1)/2$ is a prime power, then, for any positive integer m, there exists a strongly regular graph with parameters*

$$\left(\frac{2\cdot 3^{d+1}(q^{4m}-1)}{q+1}, 3^d q^{4m-1}, \frac{3^d(3^d+1)q^{4m-2}}{2}, \frac{3^d(3^d+1)q^{4m-2}}{2}\right).$$

The complement of a Spence symmetric design has parameters $(3^{d+1}(3^{d+1}-1)/2, 3^d(3^{d+1}-2), 2\cdot 3^d(3^d-1))$, and we obtain

Theorem 5.5. *Let d be a positive integer. If $q = 3^{d+1} - 2$ is a prime power, then, for any positive integer m, there exists a strongly regular graph with parameters*

$$\left(\frac{3^{d+1}(q^{4m}-1)}{2(q-1)}, 3^d q^{4m-1}, 2\cdot 3^d(3^d-1)q^{4m-2}, 2\cdot 3^d(3^d-1)q^{4m-2}\right).$$

Remark 5.6. All strongly regular graphs obtained in Theorems 5.2, 5.3, 5.4, and 5.5 are parametrically new. The smallest of these graphs has parameters $(765, 192, 48, 48)$. (Put $d = m = 1$ and $q = 3$ in Theorem 5.2.)

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