On regular quaternary Hadamard matrices

Hadi Kharaghani[*], Behruz Tayfeh-Rezaie[†], Vlad Zaitsev[‡]

Abstract

Through the use of regularizing vectors, all regular quaternary Hadamard matrices of orders 10 and 18 have been successfully identified. Of these, two matrices of order 10 and 184 matrices of order 18 were found to have unbiased mates. Converting the quaternary Hadamard matrices of order 18 to real Hadamard matrices, the study uncovered that six matrices of order 36 having unbiased pairs were connected to the extremal Pless symmetry code. Additionally, the study uncovered exactly 28 non-regular quaternary Hadamard matrices of order 18, thereby yielding the first examples of non-regular Hadamard matrices of order 36.

1 Introduction

In a recent classification of quaternary Hadamard matrices of order 18, Östergård and Paavola [9] identified 1,955,625 inequivalent matrices. We have discovered that all but 28 of these matrices are equivalent to regular ones. Furthermore, our finding indicates that out of the 1,955,625 regular matrices, only 184 have unbiased mates, and six are related to the Pless symmetry code. As a direct application of our findings, we use the 28 non-regular quaternary Hadamard matrices of order 18 to construct the first known examples of Hadamard matrices of order 36 that are not equivalent to any

[*]Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta, T1K 3M4, Canada. email: `kharaghani@uleth.ca`

[†]Institute for Research in Fundamental Sciences, P. O. Box 19395-5746, Tehran, Iran. email: `tayfeh-r@ipm.ir`

[‡]Department of Mathematics and Computer Science, University of Lethbridge, Lethbridge, Alberta, T1K 3M4, Canada. email: `vlad.zaitsev@uleth.ca`

regular matrix. The classification of quaternary Hadamard matrices of order 10 was completed by Lampio, Szöllősi and Östergård in 2013 [6]. By analyzing these smaller matrices, we found that every quaternary Hadamard matrix of order 10 is equivalent to a regular matrix.

2 Preliminaries

A *quaternary Hadamard matrix* H of order n is a square matrix with entries from the set $\{1, -1, i, -i\}$ satisfying $HH^* = nI_n$, where H^* denotes the conjugate transpose of H. Matrices with only real entries are referred to as Hadamard matrices.

A *quaternary permutation matrix* is a matrix with entries in $\{1, -1, 0, i, -i\}$ such that each row and column contains exactly one nonzero element. Two quaternary Hadamard matrices H and K are said to be (Hadamard) equivalent if there exist quaternary permutation matrices P and Q such that $H = PKQ$. A similar equivalence notion applies to standard Hadamard matrices.

Definition 1. *The sum of all entries of a Hadamard matrix $H = [H_{ij}]$ is called the* excess *of H.*

The maximum excess of a Hadamard matrix is an invariant for the matrix, and Hadamard matrices with different maximum excesses are Hadamard inequivalent, see [5] for details.

A (quaternary) Hadamard matrix is called *normalized* if all entries in its first row and first column are equal to 1. Every (quaternary) Hadamard matrix is equivalent to a normalized one. The matrix entries of a square matrix M are represented by M_{ij}.

A (quaternary) Hadamard matrix H is called *regular* if all row (or column) sums equal the same complex number. Note that when the order of the matrix is n and the row sum is s, then $n = |s|^2$. For $n = 18$, we may assume that $s = 3 + 3i$.

For a vector $v = (v_1, \ldots, v_n)$, the diagonal matrix $D = [D_{ij}]$ with entries $D_{ii} = v_i$ for $i = 1, \ldots, n$ is denoted by $\mathrm{diag}(v)$. Given an $n \times n$ quaternary Hadamard matrix H, a row vector

$$v = (v_1, \ldots, v_n) \in \{\pm 1, \pm i\}^n$$

is called a *absolutely regularizing vector* if the matrix

$$K = H\mathrm{diag}(v)$$

is *absolutely regular*. That is, there exists a constant $c > 0$ such that for all row indices i,

$$\left|\sum_{j=1}^{n} K_{ij}\right| = \left|\sum_{j=1}^{n} H_{ij}v_j\right| = c.$$

Two regularizing vectors v and w are considered equivalent if there exists a scalar $\alpha \in \{\pm 1, \pm i\}$ such that $v = \alpha w$. Unless otherwise specified, the number of regularizing vectors refers to the count of inequivalent classes under this relation.

The following property is utilized effectively for our computational searches.

Lemma 1. *Let v be a regularizing vector for a quaternary Hadamard matrix H of order* 18*. Split $H = [H_1|H_2]$, where H_1 and H_2 have nine columns, then*

$$H_1 v_1 + H_2 v_2 = (3 + 3i)u,$$

where $v = [v_1|v_2]$ and u is a vector with entries in $\{1, -1, i, -i\}$. It follows that $H_1 v_1 + H_2 v_2 = 0 \pmod 3$ in the ring of Gaussian integers modulo 3*.*

A *linear code* of length n and dimension k is a linear subspace C of $\mathbb{F}_q^n$ of dimension k. Elements of C are called *codewords*. In the case where $q = 3$, the code is referred to as a ternary code.

The *distance* between two codewords is equal to the Hamming distance between the two codewords. The *weight* of a codeword is the number of non-zero elements it has. The minimum distance of a linear code is equal to the minimum distance between two codewords, or equivalently, the minimum distance is equal to the minimum weight of its non-zero codewords. We denote a linear code of length n, dimension k and distance d as a $[n, k, d]$ code. The *dual* of a code C, denoted $C^\perp$ is defined by

$$C^\perp = \{x \in \mathbb{F}_q^n | x \cdot c = 0 \text{ for all } c \in C\}.$$

A code is said to be *self-dual* provided $C = C^\perp$. A $[n, n/2, d]$ code C over $\mathbb{F}_3$ is said to be *extremal* if C is self-dual and $d = 3\lfloor n/12 \rfloor + 3$.

The construction of extremal codes is an active research problem, having theoretical applications in combinatorics and design theory [3] and practical applications in data transmission [4].

3 Regular quaternary Hadamard Matrices of Order 18

The Hadamard matrix H is said to be of *skew-type* if $H + H^T = 2I_n$. There are approximately twenty million known Hadamard matrices of order 36, and all are known to be equivalent to regular ones [8]. A recent result [2] found that every known skew-type Hadamard matrix of order 36 is equivalent to a regular one. In an attempt to either find some Hadamard matrices of order 36 that are not equivalent to regular ones or else show that every Hadamard matrix is equivalent to a regular one, it is natural first to test the known quaternary Hadamard matrices of order 18. Denoting by $\mathcal{Q}$ the set of 1,955,625 inequivalent quaternary Hadamard matrices of order 18 in [9], we devise a fast algorithm to test the regularity of elements of $\mathcal{Q}$.

3.1 An algorithm to test regularity

For each $H \in \mathcal{Q}$, the aim is to find all vectors v with entries in $\{1, -1, i, -i\}$ having an inner product of $\pm 3 \pm 3i$ with every row of H. Thereafter, we can make H regular by multiplying the rows of $H diag(v)$ by $1, -1$, i, or $-i$. To facilitate the computation, we follow the algorithm outlined below.

The Algorithm:

(i) Split $H = [H_1|H_2]$, where H_1 and H_2 have 9 columns.

(ii) Compute $H_1 v$ and $H_2 v$ for all vectors v having entries ± 1, $\pm i$.

(iii) Find all vector pairs v, w such that $H_1 v + H_2 w = 0 \pmod 3$ in the ring of Gaussian integers modulo 3.

(iv) Concatenate each pair v, w. Define $\mathcal{R}_H$ to be the set of concatenated vectors v, w vectors with entry sum $3 + 3i$, whose inner product with every row of H is $\pm 3 \pm 3i$.

Remark 1. *Implementing the algorithm, we found all regularizing vectors for each of the* 1,955,625 *quaternary matrices of order 18. As a first result,* *we have the following.*

Theorem 1. *The outcome of the search reveals that out of* 1,955,625 *quaternary Hadamard matrices of order* 18*:*

(i) $\mathcal{R}_H = \emptyset$ *for* 28 *matrices.*

(ii) 1,955,597 *matrices are regular.*

Example: Below is an example of a non-regular Quaternary Hadamard matrix of order 18, where $-$ stands for -1 and j for $-i$.

$$Q = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & i & i & i & - & - & - & - & - & - & j & j & j \\
1 & 1 & 1 & - & - & - & 1 & j & j & i & i & - & - & - & j & 1 & 1 & i \\
1 & 1 & - & 1 & - & j & i & - & j & i & j & 1 & i & j & - & i & - & 1 \\
1 & 1 & i & - & 1 & j & j & - & i & 1 & 1 & j & - & 1 & - & - & i & - \\
1 & 1 & j & j & i & i & - & i & j & j & i & - & 1 & 1 & j & i & - & - \\
1 & i & - & - & j & 1 & 1 & i & - & 1 & j & - & 1 & - & 1 & - & j & i \\
1 & i & 1 & j & i & - & j & j & i & - & j & i & 1 & j & i & j & - & i \\
1 & i & - & - & 1 & j & - & j & j & j & - & i & i & i & i & 1 & 1 & j \\
1 & - & 1 & i & - & 1 & 1 & - & j & j & i & i & j & j & i & - & i & j \\
1 & - & i & 1 & 1 & - & - & 1 & j & 1 & i & 1 & j & - & - & - & j & i \\
1 & - & j & - & - & j & - & i & i & i & i & j & j & 1 & i & 1 & j & 1 \\
1 & - & i & - & j & i & j & i & 1 & - & 1 & 1 & i & - & j & 1 & - & j \\
1 & - & j & i & 1 & - & 1 & 1 & - & - & - & j & i & 1 & j & - & i & 1 \\
1 & - & j & 1 & j & - & i & - & i & 1 & j & i & j & i & j & 1 & i & - \\
1 & j & i & j & - & i & - & 1 & - & i & j & i & - & 1 & 1 & - & 1 & j \\
1 & j & - & i & i & i & 1 & j & 1 & - & j & j & j & i & i & i & j & - \\
1 & j & - & 1 & - & 1 & - & j & i & j & i & j & i & - & 1 & j & i & i
\end{bmatrix}.$$

The 28 matrices are available at https://www.cs.uleth.ca/ hadi/36

4 Regular Hadamard matrices of order 36

A quaternary Hadamard matrix of order $2n$ can be converted to a Hadamard matrix.

Lemma 2. *Let* H *be a quaternary Hadamard matrix of order* 18*. Split* $H = A + iB$*,* A, B $(0, \pm 1)$*-matrices of order* 18*. Then letting*

$$K = A \otimes C + B \otimes D$$

where $C = \begin{bmatrix} - & 1 \\ 1 & 1 \end{bmatrix}$ *and* $D = \begin{bmatrix} 1 & 1 \\ 1 & - \end{bmatrix}$*,* K *is a Hadamard matrix of order* 36*. Furthermore, if* H *is regular, then* K *is regular.*

As a first application, upon converting the non-regular quaternary Hadamard matrices of order 18 to Hadamard matrices, we found the first 28 examples of non-regular Hadamard matrices of order 36. The fact that only a very small fraction of quaternary Hadamard matrices of order 18 are non-regular is an indication that non-regular Hadamard matrices of order 36 are quite rare, and a reason that none were known before this work. We were able to identify 16 Hadamard inequivalent classes of non-regular Hadamard matrices of order 36 belonging to two excess inequivalent classes as follows.

- 8 equivalence classes where each matrix had a maximum excess of 204, and
- 8 equivalence classes where each matrix had a maximum excess of 208.

Remark 2. *The computation is done by splitting the matrix into two halves, each having 18 columns. For every one of the* 2^{18} *sign-flip patterns on both halves, we store all of the row sums in two separate arrays. We then loop over the* $2^{18} \times 2^{18}$ *combinations, adding the two stored row sum values for each row, taking their absolute value, and then computing the excess.*

Two examples of non-regular Hadamard matrices of order 36 are shown below for the first time. These are two examples of Hadamard matrices of order 36 with maximum excess 204 and 208, respectively.

$$H_{204} = \left[\begin{array}{cccccccccccccccccccccccccccccccccccc}
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & - & 1 \\
- & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & - & - \\
- & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & - \\
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- & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & - & - & - & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & - & - & 1 & - & 1 & - & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 \\
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1 & 1 & 1 & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - \\
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\end{array}\right]$$

$$H_{208} = \left[\begin{array}{*{36}{c}}
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 \\
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1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & - & - \\
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1 & 1 & - & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - \\
- & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - \\
1 & 1 & - & - & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & - \\
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1 & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & - \\
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\end{array}\right]$$

5 Regular quaternary Hadamard matrices of order 10

Following a similar process, we used the complete classification of quaternary Hadamard matrices of order 10 by Lampio, Szöllősi and Östergård [6], and found regularizing vectors for each of the matrices.

The complete classification of quaternary Hadamard Matrices of order 10 contains ten inequivalent matrices. Each of these ten matrices had multiple regularizing vectors.

Matrix	Number of Vectors
0	176
1	304
2	432
3	152
4	304
5	232
6	132
7	232
8	80
9	112

Table 1: Summary of computational results for order 10.

6 Unbiased quaternary Hadamard matrices of small order

Unbiased complex Hadamard matrices, linked to unbiased bases, were introduced a long time ago, see [10].

Definition 2. *Quaternary Hadamard matrices H, K of order n are called* unbiased *if $HK^* = (a+ib)L$, where L is a quaternary Hadamard matrix and $|a+ib| = \sqrt{n}$. K is called an unbiased* mate *for H.*

It is clear from the definition that each row of K is an absolutely regularizing vector for H, and n must be a sum of two squares. It is shown in [1] that the number of mutually unbiased quaternary Hadamard matrices of order $n = 2k$, k odd, is at most two. In our search for regularizing vectors, we often found many vectors, and in some cases, sufficient vectors were found to form a quaternary Hadamard matrix. This prompted us to search for all quaternary matrices of order 18 admitting an unbiased mate. We did this by adding a sixth step to our algorithm 3.1.

(v) Exhaustively search for sets of 18 mutually orthogonal vectors in $\mathcal{R}_H$.

For our search, we used the *Cliquer* program [7]. By only allowing regularizing vectors with a sum of $3+3i$, we ensure that each clique of 18 mutually orthogonal vectors results in an inequivalent unbiased mate. The 18 mutually orthogonal vectors then form the rows of the unbiased mate.

Out of 1,955,625 Quaternary Hadamard matrices, only 184 had at least one unbiased mate. The number of unbiased mates for each H ranged from 1 to 5220, as shown in Table 2 below. In total, there were 95,589 inequivalent unbiased pairs.

# of unbiased mates	**# matrices**
0	1,955,443
1	58
2	24
3	11
4	9
5	2
8	16
18	2
32	1
36	38
38	2
40	2
272	1
5184	14
5192	1
5208	1
5216	1
5220	1

Table 2: Sorting the quaternary Hadamard matrices by the number of inequivalent unbiased mates.

6.1 Application to Extremal Codes

Converting each of the 184 Quaternary Hadamard matrices with unbiased pairs to real Hadamard matrices, we obtained 6 monomially equivalent extremal codes with 3-rank 18 and minimum distance 12. Each ternary code was generated by the rows of the matrices, where the entries of the matrices were regarded as elements of $\mathbb{F}_3$.

Two inequivalent classes of quaternary Hadamard matrices were flagged, each containing three matrices. One class was recognized as *type 1* and the other of *type 2*, as described in [9]. From the two equivalence classes of Hadamard matrices, we obtain two monomially equivalent extremal codes

Table 3 shows the numbers of the codes from the 184 Quaternary Hadamard matrices with unbiased pairs. In total, there were 6 extremal codes, 68 near extremal codes and 39 self-dual codes with minimum distance 6. The remaining codes were not found to be self-dual.

(3-rank, Minimum Distance)	Count
(14, 9)	9
(14, 12)	3
(16, 6)	18
(16, 9)	40
(16, 12)	1
(18, 6)	39
(18, 9)	68
(18, 12)	6

Table 3: Count of (3-rank, minimum distance) pairs.

Each Hadamard matrix leading to an extremal code had varying numbers of mates. In the first Hadamard equivalence class, one of the three Quaternary Hadamard matrices had 272 inequivalent mates to form an unbiased pair, while the other two matrices had 18 mates respectively. When we transition to Hadamard matrices, the unbiased mates also formed extremal codes, and all 308 matrices were found to be Hadamard equivalent.

In the second Hadamard equivalence class, all three Quaternary Hadamard matrices had 1 mate respectively. As before, when we transitioned to Hadamard matrices, the mates also formed extremal codes, and the three matrices were found to be equivalent.

The six Hadamard matrices that lead to extremal codes, as well as a summary of their equivalence, can be found in Appendix A.

6.2 Order 10 unbiased quaternary Hadamard matrices

Following a similar process as for order 18, two quaternary Hadamard matrices of order 10 had unbiased mates. The result of the computation is found in the following table.

Matrix	Regularizing Vectors	Largest Clique	Unbiased Mates
0	176	6	0
1	304	6	0
2	432	10	36
3	152	4	0
4	304	6	0
5	232	5	0
6	132	5	0
7	232	5	0
8	80	4	0
9	112	10	1

Table 4: Summary of computational results for order 10.

The unbiased pair corresponding to matrix 9 can be found in Appendix B. The example corresponding to matrix 2 can be found in [1].

Acknowledgement

Hadi Kharaghani is supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

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A Extremal Codes from Unbiased Quaternary Hadamard Matrices of Order 18

Below are the quaternary Hadamard matrices of order 18 that were found to give rise to extremal codes, as well as some additional information about each matrix.

- H_1, H_2 and H_3 are Hadamard equivalent.
- H_4, H_5 and H_6 are Hadamard equivalent.
- H_1 has 272 unbiased mates.
- H_1 and H_2 have 18 unbiased mates each.
- H_4, H_5 and H_6 have 1 unbiased mate each.

$$H_1 = \left[\begin{array}{cccccccccccccccccccccccccccccccccccc}
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 \\
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1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 \\
- & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - \\
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1 & 1 & 1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & - & - \\
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- & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & - & - & - & - & - & 1 & 1 & - & - & - & 1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 \\
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- & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 \\
1 & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & - & - & 1 & 1 \\
- & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & - \\
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1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & 1 & -
\end{array}\right]$$

$$H_2 = \left[\begin{array}{cccccccccccccccccccccccccccccccccccc}
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1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
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- & 1 & - & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 \\
1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & - & - & - & 1 & 1 & - \\
- & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - \\
1 & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & - & - & - & 1
\end{array}\right]
$$

$$H_6 = \begin{bmatrix}
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & 1 & - & - & - & - & - & - & - \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & 1 & - & 1 & - & 1 \\
- & 1 & - & 1 & 1 & - & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & - & - & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & - & - & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & - & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - \\
- & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & - & 1 & - & - & - & - & 1 & - & 1 & - & 1 & - \\
1 & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & - & - \\
- & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & 1 \\
- & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 \\
1 & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 & 1 & - \\
- & 1 & - & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - & - & - & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - \\
1 & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & 1 & - & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & - & - & - & 1 \\
- & 1 & 1 & 1 & 1 & - & - & 1 & - & - & 1 & 1 & - & - & - & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 \\
1 & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & - & 1 & 1 \\
- & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - \\
1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & - & - & 1 & - & - & 1 & 1 & - & - & 1 & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & - & 1 \\
- & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & - & 1 & - & - & 1 & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & 1 & 1 & - & - & - \\
1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & - & - & - & - & 1 \\
- & 1 & 1 & - & - & 1 & 1 & 1 & 1 & 1 & - & - & - & - & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - \\
1 & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - & - & - & 1 & 1 & - & - \\
- & 1 & 1 & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - \\
1 & 1 & - & - & - & 1 & - & - & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & 1 \\
- & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & 1 & - & 1 & - & - & 1 & - & 1 & - & - & 1 & - \\
1 & 1 & - & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & - \\
- & 1 & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & - & - & 1 & - & - & - \\
1 & 1 & - & - & - & - & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & - & 1 & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & - & 1 & - & - & - & 1 \\
- & 1 & 1 & - & 1 & 1 & 1 & 1 & - & - & - & - & 1 & 1 & 1 & - & - & - & 1 & - & 1 & 1 & - & 1 & - & 1 & - & - & 1 & 1 & - & - & - & - & 1 & 1 \\
1 & 1 & - & - & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & - & - & - & 1 & - & - & 1 & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & - & 1 & - & 1 & 1 & - \\
- & 1 & 1 & - & - & - & - & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 \\
1 & 1 & - & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & - & - & - & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & 1 & 1 & - \\
- & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & - & 1 & 1 & - & - & 1 & - & - & 1 & 1 & 1 \\
1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & 1 & - & - & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & - & - & 1 & - & - & 1 & - & - & 1 & 1 & 1 & - \\
- & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 & 1 & - & - & - & - & 1 & - & 1 & - & - & 1 & 1 & - & - & - & 1 & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 \\
1 & 1 & - & 1 & 1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & 1 & - & - & - & - & 1 & 1 & - & - & - & 1 & 1 & - & 1 & - & 1 & 1 & - & - & 1 & -
\end{bmatrix}$$

B Unbiased Quaternary Hadamard Matrices of Order 10

Below is the unbiased pair corresponding to matrix 9 from Table 4.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & i & i & - & - & - & j & j \\ 1 & 1 & i & - & j & 1 & i & j & - & - \\ 1 & 1 & - & j & i & j & j & i & i & - \\ 1 & i & j & i & j & - & j & 1 & - & i \\ 1 & - & 1 & j & - & - & i & 1 & i & j \\ 1 & - & i & 1 & j & i & j & - & 1 & - \\ 1 & - & - & 1 & i & j & i & j & j & i \\ 1 & - & j & - & 1 & 1 & - & - & i & 1 \\ 1 & j & - & - & - & i & 1 & i & j & 1 \end{bmatrix}$$

$$K = \begin{bmatrix} 1 & 1 & 1 & j & i & 1 & - & j & 1 & - \\ 1 & j & - & 1 & 1 & i & j & 1 & j & i \\ 1 & i & j & 1 & 1 & j & i & 1 & - & j \\ j & 1 & 1 & 1 & j & - & - & i & 1 & 1 \\ j & 1 & 1 & i & i & i & 1 & j & - & 1 \\ j & - & 1 & i & j & 1 & 1 & 1 & 1 & - \\ j & i & - & j & i & 1 & 1 & i & 1 & 1 \\ i & 1 & 1 & j & j & 1 & 1 & i & - & i \\ i & 1 & - & i & j & 1 & i & j & 1 & 1 \\ i & - & 1 & j & 1 & - & 1 & j & 1 & 1 \end{bmatrix}$$