Singular Value Decomposition

For m by n matrix A there is a unitary m by m matrix U and a unitary n by n matrix V with

$$A = USV^*$$

where $S = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$, and $D = \text{diag}(\mu_1, \ldots, \mu_r)$ with $\mu_1 \geq \cdots \geq \mu_r > 0$.

Proof:

Let $\mathbf{v}_1, \cdots, \mathbf{v}_n$ be an orthonormal basis of eigenvectors for A^*A. This matrix is Hermitian so such a basis exists and the eigenvalues are real; so arrange the eigenvectors so that the eigenvalues are in descending order.

$$A^*A\mathbf{v}_i = \lambda_i \mathbf{v}_i.$$

The eigenvalues λ_i are non-negative because:
$\langle \mathbf{v}_i, \mathbf{v}_i \rangle > 0$ ($\mathbf{v}_i \neq \mathbf{0}$ for eigenvectors) and $\lambda_i \langle \mathbf{v}_i, \mathbf{v}_i \rangle = \langle \mathbf{v}_i, \lambda_i \mathbf{v}_i \rangle = \langle \mathbf{v}_i, A^*A\mathbf{v}_i \rangle = \langle A\mathbf{v}_i, A\mathbf{v}_i \rangle \geq 0$.

Also if $\lambda_i = 0$ then this shows $\langle A\mathbf{v}_i, A\mathbf{v}_i \rangle = 0$ so $A\mathbf{v}_i = 0$.

Let

$$\mathbf{u}_i = \frac{1}{\sqrt{\lambda_i}} A\mathbf{v}_i$$

for $\lambda_i \neq 0$. These are orthonormal since

$$\begin{aligned}
\langle \mathbf{u}_i, \mathbf{u}_j \rangle &= \frac{1}{\sqrt{\lambda_i}} \frac{1}{\sqrt{\lambda_j}} \langle A\mathbf{v}_i, A\mathbf{v}_j \rangle \\
&= \frac{1}{\sqrt{\lambda_i}} \frac{1}{\sqrt{\lambda_j}} \langle \mathbf{v}_i, A^*A\mathbf{v}_j \rangle \\
&= \frac{1}{\sqrt{\lambda_i}} \frac{1}{\sqrt{\lambda_j}} \langle \mathbf{v}_i, \lambda_j \mathbf{v}_j \rangle \\
&= \frac{1}{\sqrt{\lambda_i}} \frac{\lambda_j}{\sqrt{\lambda_j}} \langle \mathbf{v}_i, \mathbf{v}_j \rangle
\end{aligned}$$

where the last term is 0 if $i \neq j$ and 1 if $i = j$.

Extend the $\mathbf{u}_i$'s to an orthonormal basis of C^m.

Let $U = (\mathbf{u}_1 | \ldots | \mathbf{u}_m)$ and $V = (\mathbf{v}_1 | \ldots | \mathbf{v}_n)$.

Then

$$AV\mathbf{e}_i = A\mathbf{v}_i = \begin{cases} \sqrt{\lambda_i}\mathbf{u}_i & \text{if } \lambda_i \neq 0 \\ 0 & \text{if } \lambda_i = 0 \end{cases} = \sqrt{\lambda_i}\mathbf{u}_i = U \begin{bmatrix} \sqrt{\lambda_1} & 0 & \ldots \\ 0 & \ddots & \ldots \\ \vdots & \vdots & \vdots \end{bmatrix} \mathbf{e}_i = US\mathbf{e}_i,$$

where $\mu_i = \sqrt{\lambda_i}$, so $AV = US$.