Dimension of Bases

DIMENSION THEOREM: All bases for a vector space V have the same number of vectors. This number, denoted $\dim(V)$, is called the *dimension* of V.

LEMMA: No proper (strict) subset of a basis is a basis.

PROOF: Suppose, say, $\{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$ and $\{\mathbf{u}_2, \ldots\}$ are both bases. Since $\{\mathbf{u}_2, \ldots\}$ spans, we see $\mathbf{u}_1$ is a lin. comb. of $\{\mathbf{u}_2, \ldots\}$. Write $\mathbf{u}_1 = c_2\mathbf{u}_2 + \cdots$. Then $\mathbf{0} = -1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots$ contradicting lin. indep. of $\{\mathbf{u}_1, \mathbf{u}_2, \ldots\}$.

PROPOSITION: If $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ is a basis and $\mathbf{v}_1 = c_1\mathbf{w}_1 + \cdots$ with $c_1 \neq 0$ then $\{\mathbf{v}_1, \mathbf{w}_2, \ldots\}$ is also a basis.

PROOF: *Lin. indep.:* If $a_1\mathbf{v}_1 + a_2\mathbf{w}_2 + \cdots = \mathbf{0}$ then replacing $\mathbf{v}_1$ gives $a_1(c_1\mathbf{w}_1 + \cdots) + a_2\mathbf{w}_2 + \cdots = \mathbf{0}$. The coefficient of $\mathbf{w}_1$ is a_1c_1 so since $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ is lin. indep. we have $a_1c_1 = 0$. But $c_1 \neq 0$ so $a_1 = 0$. Thus $a_2\mathbf{w}_2 + \cdots = \mathbf{0}$ and again by lin. indep. of $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$, the rest of the a_i's are zero.
Span: Given $\mathbf{u}$, since $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ span write $\mathbf{u} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots$. Now $c_1\mathbf{w}_1 = \mathbf{v}_1 - c_2\mathbf{w}_2 - \cdots$, so $\mathbf{u} = a_1c_1^{-1}(\mathbf{v}_1 - c_2\mathbf{w}_2 + \cdots) + a_2\mathbf{w}_2 + \cdots$, which is a lin. comb. of $\{\mathbf{v}_1, \mathbf{w}_2, \ldots\}$.

STEINITZ REPLACEMENT THEOREM: If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ are different bases for a vector space, then (after relabeling) there are vectors $\mathbf{v}_1 \notin \{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ and $\mathbf{w}_1 \notin \{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ such that $\{\mathbf{v}_1, \mathbf{w}_2, \ldots\}$ is also a basis for the vector space.

PROOF: Since the bases are different and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ cannot be a proper subset of $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ by the lemma, we may assume (after relabeling of $\mathbf{v}$'s) that $\mathbf{v}_1 \notin \{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$. Since $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ span, write $\mathbf{v}_1 = c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$ where $c_i \neq 0$ (which may require relabeling of $\mathbf{w}$'s). Now if $\mathbf{w}_1, \ldots, \mathbf{w}_n$ were all in $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ then $\mathbf{0} = -1\mathbf{v}_1 + c_1\mathbf{w}_1 + \cdots + c_n\mathbf{w}_n$ which contradicts that $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ is a basis — recall $\mathbf{v}_1 \neq \mathbf{w}_i$. Thus (after further relabeling of the $\mathbf{w}$'s) we may assume $\mathbf{w}_1 \notin \{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$. The Proposition completes the proof.

PROOF OF DIMENSION THEOREM: Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ and $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ be two different bases. Apply the Steinitz Replacement Theorem, noting that $\{\mathbf{v}_1, \mathbf{w}_2, \ldots\}$ has the same number of vectors as $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ and that $\{\mathbf{v}_1, \mathbf{w}_2, \ldots\}$ has more vectors in common with $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ than $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ has.
By repeated use of the Steinitz Replacement Theorem we can transform the second basis so that it always has the same number of vectors as $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ while making the intersection with $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ progressively larger. This process can only stop when the second basis is $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$. Thus $\{\mathbf{w}_1, \mathbf{w}_2, \ldots\}$ and $\{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ have the same number of elements.

REMARK FOR INFINITE CASE: A similar proof works in the infinite case. However, the Steinitz Replacement Theorem must be used an infinite number of times. Each use of it requires an arbitrary choice (namely, that of the vector to label as $\mathbf{v}_1$). An infinite number of arbitrary choices cannot be made without a special axiom: the axiom required is *The Axiom of Choice.*