

## A NOTE ON DERANGEMENTS

There is a somewhat complicated formula obtained via inclusion-exclusion for the number  $d_n$  of derangements of  $1, 2, \dots, n$ .

Alternately it is possible to derive the recurrence relation

$$d_n = (n - 1)(d_{n-1} + d_{n-2})$$

with  $d_2 = 1, d_1 = 0$  or perhaps better, with  $d_1 = 0, d_0 = 1$ .

This recurrence relations can be rewritten in a better form. Bringing  $nd_{n-1}$  to the left gives:

$$d_n - nd_{n-1} = -d_{n-1} + (n - 1)d_{n-2}.$$

Multiply by  $(-1)^n$  gives:

$$(-1)^n(d_n - nd_{n-1}) = (-1)^{n-1}(d_{n-1} - (n - 1)d_{n-2}).$$

Now iterate this formula, so that ultimately on the right when  $n$  is reduced to 3 we obtain:

$$= (-1)^2(d_2 - (2)d_1) = 1.$$

Thus:

$$(-1)^n(d_n - nd_{n-1}) = 1$$

Solving for  $d_n$  gives a recurrence relation, which is much better than the first one above:

$$d_n = nd_{n-1} + (-1)^n$$

with  $d_1 = 0$  or better yet with  $d_0 = 1$ .

This recurrence is easy to implement and is strikingly similar to the one for  $f_n = n!$ , namely:

$$f_n = nf_{n-1}, \quad f_0 = 1$$