Proof of the Product Rule

Suppose f and g are differentiable at x. Then there are numbers $Df(x)$ and $Dg(x)$ with

$$f(x+h) = f(x) + Df(x)h + R_f h$$
$$g(x+h) = g(x) + Dg(x)h + R_g h$$

where

$$R_f \to 0 \quad \text{as } h \to 0$$
$$R_g \to 0 \quad \text{as } h \to 0$$

Multiply these equations . . .

Proof — equations multiplied

$$f(x+h)\cdot g(x+h)$$

$$= [f(x)+Df(x)h+R_f h]\cdot[g(x)+Dg(x)h+R_g h]$$

$$\overset{see}{\underset{table}{=}} \boxed{f(x)g(x)} + \boxed{[Df(x)g(x) + f(x)Dg(x)]\,h} + Rh$$

	$g(x)$	$Dg(x)h$	$R_g h$
$f(x)$	$\boxed{f(x)g(x)}$	$\boxed{f(x)Dg(x)h}$	$f(x)R_g h$
$Df(x)h$	$\boxed{Df(x)g(x)h}$	$Df(x)hDg(x)h$	$Df(x)hR_g h$
$R_f h$	$R_f g(x)h$	$R_f hDg(x)h$	$R_f hR_g h$

Proof concluded

We have

$$f(x+h)g(x+h)$$
$$= f(x)g(x) + [Df(x)g(x) + f(x)Dg(x)]\,h + Rh$$

where R involves terms with at least one R_f, R_g or h and so $R \longrightarrow 0$ as $h \longrightarrow 0$.
Therefore the derivative of $f(x)g(x)$ is the term

$$Df(x)g(x) + f(x)Dg(x).$$