Differentiation Rules (Differential Calculus)

1. Notation

The derivative of a function f with respect to one independent variable (usually x or t) is a function that will be denoted by Df. Note that $f(x)$ and $(Df)(x)$ are the values of these functions at x.

2. Alternate Notations for $(Df)(x)$

For functions f in one variable, x, alternate notations are: $D_x f(x)$, $\frac{d}{dx}f(x)$, $\frac{df(x)}{dx}$, $\frac{df}{dx}(x)$, $f'(x)$, $f^{(1)}(x)$. The "(x)" part might be dropped although technically this changes the meaning: f is the name of a function, whereas $f(x)$ is the value of it at x. If $y = f(x)$, then $D_x y$, $\frac{dy}{dx}$, y', etc. can be used. If the variable t represents time then $D_t f$ can be written $\dot{f}$. The differential, "df", and the change in f, "Δf", are related to the derivative but have special meanings and are never used to indicate ordinary differentiation.
Historical note: Newton used $\dot{y}$, while Leibniz used $\frac{dy}{dx}$. About a century later Lagrange introduced y' and Arbogast introduced the operator notation D.

3. Domains

The domain of Df is always a subset of the domain of f. The conventional domain of f, if $f(x)$ is given by an algebraic expression, is all values of x for which the expression is defined and results in a real number. If f has the conventional domain, then Df usually, but not always, has conventional domain. Exceptions are noted below.

4. Operating Principle

Many functions are formed by successively combining simple functions, using constructions such as sum, product and composition. To differentiate, apply the differentiation rule corresponding to the *last* construction.

5. Rules for Constructions

SUM:	$D(f+g) = Df + Dg$	
LINEARITY:	$D(af+bg) = aDf + bDg$	where a and b constant.
PRODUCT:	$D(f \cdot g) = Df \cdot g + f \cdot Dg$	
RECIPROCAL:	$D(1/f) = -Df/f^2$	
QUOTIENT:	$D(f/g) = \dfrac{g \cdot Df - f \cdot Dg}{g^2}$	
CHAIN (for compositions):	$D(f \circ g) = (Df) \circ g \cdot Dg$	
	$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$	where $u = g(x)$ and $y = f(u) = f(g(x))$.

INVERSE:	$D(\operatorname{inv}(f)) = 1 \Big/ (Df \circ \operatorname{inv}(f))$	where $\operatorname{inv}(f)$ is the inverse function to f.
	$D_y g = 1/D_x f$	where $y = f(x)$ and $x = g(y)$.
LOGARITHMIC DIFF:	$D(f) = f \cdot D(\ln\|f\|)$	but simplify $\ln\|f\|$ first.
FUNDAMENTAL TH.:	$D\int_a^x f\,dx = f(x)$	if f is continuous at x.

6. Rules in special situations

On expressions like $k \cdot f(x)$ where k is constant do not use the product rule — use linearity.
On expressions like $1/f(x)$ do not use quotient rule — use the reciprocal rule, that is, rewrite this as $f(x)^{-1}$ and use the Chain rule.
Use logarithmic differentiation to avoid product and quotient rules on complicated products and quotients and also use it to differentiate powers that are messy.
One can use $b^p = e^{p\ln b}$ to differentiate powers.
Use $\log_b|x| = \ln|x|/\ln b$ to differentiate logs to other bases.

7. Rules for Elementary Functions

$Dc = 0$	where c is constant.
$D(ax+b) = a$	where a and b are constant.
$Dx^p = px^{p-1}$	p constant. For non-constant p use logarithmic diff. or rewrite as $e^{p\ln x}$.
$Dx = 1$	This is $p = 1$ in above.
$Dx^2 = 2x$	This is $p = 2$ in above.
$D\sqrt{x} = \dfrac{1}{2\sqrt{x}}$	This is $p = 1/2$ in above.
$D1/x = -1/x^2$	This is $p = -1$ in above.
$D\|x\| = x/\|x\| = \operatorname{sgn}(x)$	Domain of derivative: $x \neq 0$. Note: $\|x\| = \sqrt{x^2}$
$D\operatorname{sgn}(x) = 0$	Domain of derivative: $x \neq 0$.
$D\lfloor x\rfloor = D\lceil x\rceil = 0$	Domain of derivative: x not an integer (0, ±1, ±2, $\ldots$).
$D\sin x = \cos x$	
$D\cos x = -\sin x$	(derivatives of cotrigs always have minus signs.)
$D\tan x = \sec^2 x = 1 + \tan^2 x$	
$D\sec x = \sec x \tan x$	
$De^x = e^x$	
$D\ln x = 1/x$	Domain of derivative: $x > 0$.
$D\ln\|x\| = 1/x$	
$D\arctan x = 1/(1+x^2)$	
$D\arcsin x = 1/\sqrt{1-x^2}$	
$D\arccos x = -1/\sqrt{1-x^2}$	

$D\cosh x = \sinh x$
$D\sinh x = \cosh x$
$D\tanh x = \operatorname{sech}^2 x = 1 - \tanh^2 x$
$D\operatorname{sech} x = -\operatorname{sech} x \tanh x$
$D\operatorname{arctanh} x = 1/(1-x^2)$
$D\operatorname{arccosh} x = 1/\sqrt{-1+x^2}$ Domain of derivative: $x > 1$.
$D\operatorname{arcsinh} x = 1/\sqrt{1+x^2}$

8. Trig Tricks

If the only occurrences of x in a trig identity are of form trig(x) or as $D = D_x$, then break the identity down to sin's and cos's and make the following replacements to get another valid identity:
co-trick: Replace "cos" by "$-$sin" and "sin" by "cos". This amounts to replacing x by $x + \pi/2$.
i-*trick:* Replace "cos" by "cosh", "sin" by "$\mathrm{i}\sinh$" and "D" by "$-\mathrm{i}D$" where $\mathrm{i}^2 = -1$.
Also keep $\cos^2 x + \sin^2 x = 1$ and $\cosh^2 x + \sinh^2 x = 1$ and the addition formulas in mind since they are especially useful when antidifferentiating.

9. Evaluation after Differentiation

To evaluate Df at a particular number, say $x = 17$, use the evaluation bar notation: "$\big|_{x=17}$" or if the variable is clear use "$\big|_{17}$", thus we have $D_x f(x)\,\big|_{17}$, $\frac{dy}{dx}\,\big|_{x=17}$, etc. You can also use $\frac{df}{dx}(17)$, $f'(17)$, $f^{(1)}(17)$, $y'(17)$, each of which say to differentiate first and then evaluate at 17 afterward. Notations such as $D_x(f(17))$ or $\frac{d}{dx}f(17)$ are not used because both equal 0 since the evaluation is done first; this is usually not what was intended.

10. Higher Order Derivatives

$D^3 f(x)$, $D_x^3 f(x)$, $\frac{d^3}{dx^3} f(x)$, $f'''(x)$, $f^{(3)}(x)$, $\frac{d^3 y}{dx^3}$, etc. mean differentiate three times. Similarly for differentiating 0, 1, 2, 3, 4, ... times.

11. Differentiation of Functions in more than One Variable

Consider a function in three variables with value $f(x,y,z)$.
If y and z depend on x then use above notations, but in order to compute the derivative implicit differentiation, that is, the chain rule must be used.
If x, y and z are independent variables then a derivative can be computed by treating y and z as constants and differentiating with respect to x. This derivative is called a *partial derivative* and is denoted by $\frac{\partial}{\partial x} f$, $D_1 f$, $D_x f$, f_x or similarly. For example,

$$D_{z,x} x^2 y^3 z^4 = \frac{\partial}{\partial z}\frac{\partial}{\partial x} x^2 y^3 z^4 = \frac{\partial}{\partial z} 2xy^3 z^4 = 2xy^3 4z^3.$$