Distribution Problems

The property that characterizes a distribution (occupancy) problem is that a ball (object) must go into **exactly one** box (bin or cell). This amounts to a function from balls to bins.

	n distinguishable boxes		n indistinguishable boxes	
	empty box allowed	no box empty	empty box allowed	no box empty
r disting. balls	n^r	$n!\left\{{r \atop n}\right\}$	$\sum_{i=1}^{n}\left\{{r \atop i}\right\}$	$\left\{{r \atop n}\right\}$
r indisting. balls	$\binom{r+n-1}{r}$	$\binom{r-1}{n-1}$	$\sum_{i=1}^{n}\left\|{r \atop i}\right\|$	$\left\|{r \atop n}\right\|$

We now explain the entries working from right to left.

$\left\{{r \atop n}\right\}$: This is *by definition.* It is the same as the number of n-subsets of r balls.

$\sum_{i=1}^{n}\left\{{r \atop i}\right\}$: Use the previous and the addition principle on the cases: r balls in 1 box none empty, r balls into 2 boxes none empty, etc.

$n!\left\{{r \atop n}\right\}$: Put the balls into indistinguishable boxes ($\left\{{r \atop n}\right\}$ ways). The boxes are now distinguishable by their contents. Then put labels on the boxes ($n!$ ways). Later we show by inclusion-exclusion that:

$$n!\left\{{r \atop n}\right\} = \sum_{i=0}^{n}(-1)^i\binom{n}{i}(n-i)^r$$

which provides an alternative to the double recursion formula below for computing $\left\{{r \atop n}\right\}$.

n^r: Just an r-sequence — for each ball there are n ways to put it in a box.

$\left|{r \atop n}\right|$: This is *by definition.* It is the number of partitions of r into n parts, that is, write r as a sum of natural numbers, order unimportant. For example, for $r = 4$, $n = 2$ the partitions are: $4 = 3 + 1$ and $4 = 2 + 2$. Thus $\left|{4 \atop 2}\right| = 2$. The partition $3 + 1$ says put 3 balls in one box and 1 in the other.

$\sum_{i=1}^{n}\left|{r \atop i}\right|$: Use the previous and the addition principle on the cases: r balls in 1 box none empty, r balls into 2 boxes none empty, etc. It is the number of partitions of r into n *or fewer* parts. For example, 4 has partitions 4, $3 + 1$, $2 + 2$, $2 + 1 + 1$, $1 + 1 + 1 + 1$.

$\binom{r+n-1}{r} = \binom{r+n-1}{n-1}$: This is an arrangement of r balls and $n - 1$ dividers. Choose the positions for the balls or the dividers, whichever you prefer.

$\binom{r-1}{n-1}$: First take n balls and put one ball in each box. This leaves $r - n$ balls to distribute with no restrictions — the previous case. Thus there are $\binom{(r-n)+n-1}{n-1}$ ways.

Double recurrence relations

These can be used to compute the above quantities. Proofs are provided elsewhere.

Choose: $C(r, n) = \binom{r}{n}$ has recurrence $\binom{r}{n} = \binom{r-1}{n} + \binom{r-1}{n-1}$ called Pascal's formula.

Partition: $p(r, n) = \left|{r \atop n}\right|$ has recurrence $\left|{r \atop n}\right| = \left|{r-n \atop n}\right| + \left|{r-1 \atop n-1}\right|$.

Subset: $S(r, n) = \left\{{r \atop n}\right\}$ has recurrence $\left\{{r \atop n}\right\} = n\left\{{r-1 \atop n}\right\} + \left\{{r-1 \atop n-1}\right\}$.

Cycle (defined below): $c(r, n) = s(r, n) = \left[{r \atop n}\right]$ has recurrence $\left[{r \atop n}\right] = (r-1)\left[{r-1 \atop n}\right] + \left[{r-1 \atop n-1}\right]$.

The boundary (initial) conditions are that each is 0 if either $n = 0$ or $r = 0$ but 1 if both $n = 0$ and $r = 0$, with the exception that $\binom{r}{0} = 1$ regardless of r. Also $\left|{s \atop n}\right| = 0$ if $s < 0$, a required initial condition since $r - n$ above could be negative.

$\left\{{r \atop n}\right\}$ is called a Stirling number of the second kind while $\left[{r \atop n}\right]$ is called a Stirling number of the first kind.

$\left[{r \atop n}\right]$ is the number of permutations of r objects that have n cycles. For example, permuting 1-2-3-4-5 to 2-4-5-1-3 takes the 1-st object to 4-th position, 4-th to 2-nd and 2-nd to 1-st (and so 1, 4, 2 is called a cycle). Also it takes the 3-rd object to 5-th position and 5-th to the 3-rd (forming a second cycle 3, 5). This permutation is said to have two cycles. 1-2-3-4-5 to 2-3-4-1-5 also has two cycles. There are actually 50 5-permutations that have 2 cycles, so $\left[{5 \atop 2}\right] = 50$.