Duality for Standard Linear Programming Problems

Throughout, the non-negativity constraints are assumed but suppressed.

PRIMAL STANDARD LP PROBLEM:

$$\begin{array}{ll} \text{Maximize} & 51x_1 + 52x_2 - 53x_3 + 50 \quad (= z) \\ \text{Subject to} & 11x_1 + 12x_2 + 13x_3 \le 10 \\ & 21x_1 + 22x_2 + 23x_3 \le 20 \end{array}$$

DUAL PROBLEM:

$$\begin{array}{ll} \text{Minimize} & 10y_1 + 20y_2 + 50 \quad (= w) \\ \text{Subject to} & 11y_1 + 21y_2 \ge 51 \\ & 12y_1 + 22y_2 \ge 52 \\ & 13y_1 + 23y_2 \ge -53 \end{array}$$

DUAL PROBLEM IN STANDARD FORM:

$$\begin{array}{ll} \text{Maximize} & -10y_1 - 20y_2 - 50 \quad (= -w) \\ \text{Subject to} & -11y_1 - 21y_2 \le -51 \\ & -12y_1 - 22y_2 \le -52 \\ & -13y_1 - 23y_2 \le 53 \end{array}$$

PRIMAL DICTIONARY:[1]

$$\begin{aligned} s_1 &= -11x_1 - 12x_2 - 13x_3 + 10 \\ s_2 &= -21x_1 - 22x_2 - 23x_3 + 20 \\ z &= 51x_1 + 52x_2 - 55x_3 + 50 \end{aligned}$$

DUAL DICTIONARY:

$$\begin{aligned} t_1 &= 11y_1 + 21y_2 - 51 \\ t_2 &= 12y_1 + 22y_2 - 52 \\ t_3 &= 13y_1 + 23y_2 + 53 \\ -w &= -10y_1 - 20y_2 - 50 \end{aligned}$$

POSSIBLE SIMPLEX TABLEAU[2] FOR PRIMAL DICTIONARY:

$$A = \begin{array}{r|rrrrrr} & & & x_1: & x_2: & x_3: & 1: \\ s_1: & -1 & 0 & -11 & -12 & -13 & 10 \\ s_2: & 0 & -1 & -21 & -22 & -23 & 20 \\ z: & 0 & 0 & 51 & 52 & -53 & 50 \end{array}$$

ASSOCIATED DUAL SIMPLEX TABLEAU:

$$-A^T = \begin{array}{r|rrr} & y_1: & y_2: & 1: \\ & 1 & 0 & 0 \\ & 0 & 1 & 0 \\ t_1: & 11 & 21 & -51 \\ t_2: & 12 & 22 & -52 \\ t_3: & 13 & 23 & 53 \\ -w: & -10 & -20 & -50 \end{array}$$

[1] For clarity, use slack variables s_i instead of x_{n+i}. Also write the constant terms, as is conventional for systems of equations, last instead of first.

[2] In this version of tableaux, the main body has the same coefficients as the dictionary does! Strictly speaking, the tableau does not record which variables are to be considered basic although the -1 entries give a hint; this information should be carried along with the tableau.

Correspondences between variables:

primal constraints/slack variables s_i	$\longleftrightarrow$	dual decision variables y_i
primal decision variables x_j	$\longleftrightarrow$	dual constraints/slack variables t_j

Note that free variables (ones with no non-negativity constraints) in one problem correspond to equality constraints in the other problem and visa versa. This follows easily from the following substitution which allows the arguments given below to extend to the case of free variables and equality constraints.
Replace each equality constraint $\sum a_{ij}x_j = b_i$ by two constraints $\sum a_{ij}x_j \leq b_i$ and $\sum -a_{ij}x_j \leq -b_i$. In the dual problem, this gives rise to two nonnegative variables yp_i and ym_i whose coefficients have opposite signs. Combining as $y_i = yp_i - ym_i$, we obtain a dual variable which is free.
Each free variable x_j can be written as the difference $xp_j - xm_j$ of two nonnegative variables xp_j and xm_j. In the dual problem this gives rise to two inequality constraints which combine into the equality constraint $\sum a_{ij}y_i = c_j$.

A trick to avoid the 2-phase simplex method. The 2-phase simplex method can be avoided by working with a modified LP problem. Introduce the variable x_0 obtaining:

$$\begin{array}{lrcrcl} \text{Maximize} & c_j x_j & - & M x_0 & & (= z) \\ \text{Subject to} & \sum a_{ij} x_j & - & x_0 & \leq & b_i \end{array}$$

where M is a symbol with $M \gg 1$. Even better treat $aM + b$ lexicographically with $M > b$ for every b. As a first step make x_0 basic and s_I nonbasic, where b_I is the least of the b_i's. The modified problem is now feasible. Use the 1-phase simplex method, but make x_0 nonbasic at the first chance possible. Once nonbasic, x_0 and M can be ignored. If x_0 remains basic, then the original problem is infeasible. The original tableau will contain an extra column

$$\begin{array}{c} x_0: \\ +1 \\ \vdots \\ +1 \\ -M \end{array}$$

while the dual tableau contains an extra row

$$t_0: \quad -1 \quad \cdots \quad -1 \quad +M$$

which corresponds to an extra constraint $-y_1 - \cdots - y_m + M \geq 0$, that is, $y_1 + \cdots + y_m \leq M$. Note that this constraint effectively makes the modified dual problem bounded. Thus feasibility and boundedness appear dual to each other: If the primal problem is feasible then the dual problem can not be unbounded, that is, is infeasible or optimal.

The key observation. In solving equations one usually uses row operations (changing equations) but one could use column operations (changing variables) instead.
The main step in the simplex method for the primal problem is to shift around the columns containing the single -1 entries in the tableau, the "basic" columns. This is done by pivoting,[3] that is, by a sequence of row operations whose net effect can be recorded by a matrix R. Thus $A\mathbf{x} = \mathbf{0}$ becomes $B\mathbf{x} = \mathbf{0}$ where $B = RA$. Taking negative transposes gives $-B^T = (-A^T)C$ where $C = R^T$. Thus applying the corresponding sequence of column operations, shifts around the rows containing the single 1 entries in the dual tableau

[3] On the negative transpose matrix these gives rise to a series of steps which amount to the "dual simplex method" if the original dual problem is viewed as a new primal problem.

while still preserving the negative transpose relationship between tableaux. Moreover, $(-A^T)\mathbf{y} = \mathbf{0}$ is $(-A^T)C \quad C^{-1}\mathbf{y} = \mathbf{0}$. So we obtain $-B^T\mathbf{y}' = \mathbf{0}$, a new set of equations in the new variables[4] $\mathbf{y}' = C^{-1}\mathbf{y}$. We claim, however, that these new variables, are really just some of the y_i or t_j. The example can be used to illustrate the general argument. Suppose the 1 0 0 and 0 1 0 rows have moved to the 4th and 1st rows respectively in $-B^T$. Then[5]

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \\ 1 \end{pmatrix} = \begin{pmatrix} \text{4th row of } -B^T \\ \text{1st row of } -B^T \end{pmatrix} \begin{pmatrix} y_1' \\ y_2' \\ 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \text{4th row of } -A^T \\ \text{1st row of } -A^T \end{pmatrix} C \begin{pmatrix} y_1' \\ y_2' \\ 1 \end{pmatrix} = \begin{pmatrix} 12 & 22 & -52 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ 1 \end{pmatrix} = \begin{pmatrix} t_2 \\ y_1 \end{pmatrix}$$

Thus the change of variables merely amounts to rewriting the equations in terms of some other subset of the original variables. Such a change can also be achieved by pivoting (row operations!) on the dual dictionary. Therefore, a change done to the primal dictionary/tableau by using the simplex method can be mimicked on the dual dictionary so that respective dictionaries/tableaux remain negative transposes and both still represent the same solutions to their respective systems.

Consequences for optimality. The general form of the coefficients of an **optimal** primal dictionary is (since it is basic):

$$\begin{array}{rcc} & x_j': & 1: \\ s_i': & a_{ij}' & +b_i' \\ z: & -c_j' & v \end{array}$$

where $+b_i' \geq 0$ by feasibility and $-c_j' \leq 0$ by maximality. Then the coefficients of the dual dictionary have form

$$\begin{array}{rcc} & y_i': & 1: \\ t_j': & -a_{ji}' & +c_j'. \\ -w: & -b_i' & -v \end{array}$$

Thus the dual dictionary is also optimal because from its form we get a basic solution (namely let $y_i' = 0$), which is feasible because $+c_j' \geq 0$ and maximal since $-b_i' \leq 0$. Moreover, its maximum is $-w = -v$, the negative of the primal maximum $z = v$. This proves:

DUALITY THEOREM

If the primal problem has an optimum then the dual does. The optimal w for the dual problem is the optimal z for the primal problem, both being v in the optimal primal dictionary. An optimum solution is $x_j' = 0$, $s_i' = b_i'$, $y_i' = 0$ and $t_j' = c_j'$. The s_i', x_j' are a rearrangement of s_i, x_j while the y_i', t_j' are the corresponding rearrangement of y_i, t_j.

Further, these values can be read from just the primal optimal dictionary.

Example: If the final primal dictionary is

$s_2 \;=\; 711s_1 - 712x_2 + 713x_3 + 1000$

$x_1 \;=\; 721s_1 + 722x_2 - 723x_3 + 2000$

$z \;=\; -501s_1 - 502x_2 - 503x_3 + 888.$

then the optimal z and w value is 888 achieved by:

$(s_1, x_2, x_3, s_2, x_1) = (0, 0, 0, 1000, 2000)$ for the primal problem and

$(y_1, t_2, t_3, y_2, t_1) = (501, 502, 503, 0, 0)$ for the dual problem.

Thus $x_1 = 2000$, $x_2 = x_3 = 0$ for the primal and $y_1 = 501$, $y_2 = 0$ for the dual problems.

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[4] The last entry still represents the constant 1 for the following reason. R has last column the transpose of $00\cdots1$, because in the simplex method the last row of the tableau is never added to any other row. Thus C, and so also C^{-1}, has last row $00\cdots1$.

[5] C is a 3 by 3 matrix.